Cornish-fisher Expension for Commercial, Real Estate Value at Risk

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Cahier du CEMOTEV n° 2014-02

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February 8, 2014

Abstract

The computation of Value at Risk has traditionally been a troublesome issue in commercial real estate. Difficulties mainly arise from the lack of appropriate data, the non-normality of returns, and the inapplicability of many of the traditional methodologies. As a result, calculation of this risk measure has rarely been done in the Real Estate field. However, following a spate of new regulations such as Basel II, Basel III, NAIC and Solvency II, financial institutions have increasingly been required to estimate and control their exposure to market risk. As a result, financial institutions now commonly use "internal" Value at Risk (VaR) models in order to assess their market risk exposure. The purpose of this paper is to estimate distribution functions of real estate VaR while taking into account non-normality in the distribution of returns. This is accomplished by the combination of the Cornish-Fisher expansion with a certain rearrangement procedure. We demonstrate that this combination allows superior estimation, and thus a better VaR estimate, than has previously been obtainable. We also show how the use of our rearrangement procedure solves well-known issues arising from the monotonicity assumption required for the Cornish-Fisher expansion to be applicable, a difficulty which has previously limited the useful of this expansion technique. Thus, practitioners can find a methodology here to quickly assess value at risk without suffering loss of relevancy due to any non-normality in their actual return distribution. The originality of this paper lies in our particular combination of Cornish-Fisher expansions and the rearrangement procedure.

Keywords: Value at Risk, Risk Measurement, Real Estate Finance, Cornish-Fisher Expansion, Risk Management, Rearrangement Procedures

1 Introduction

The stock market crash of 1987 triggered the development of new risk measures. This was the first major financial crisis in which practitioners and academicians became concerned with the possibility of global bankruptcy. The crash was so improbable given then standard statistical models that quantitative analysts began to question the appropriateness of their techniques. Numerous academics, proclaiming that the crisis could easily reoccur, called for reconsideration of all such risk models. It had become obvious that the possibility of extreme events occurring required much greater examination. Limitations of traditional risk measures were acknowledged, with improved measurement of risks for a major decline in asset value having become an urgent task. There was a recognized need for greater reliance on a risk measure considering the entire return distribution of a portfolio. During the 1990s, a comprehensive new risk measure did emerge: Value at Risk, commonly known by the acronym VaR. Practitioners and regulators developed and then increasingly adopted the VaR measure in their subsequent risk analyses.

Informally, value at risk is the largest percentage loss with a given probability (confidence level) likely to be suffered on a portfolio position over a given holding period. In other words, for a given portfolio and time horizon, and having selected a confidence level, $\alpha \in (0, 1)$, VaR is defined to be that threshold value, assuming no further trade, such that the probability that the mark-to-market loss in the portfolio exceeds this VaR level is exactly the preset probability of loss α.[1] Thus, VaR is the quantile of the projected distribution of losses over the target horizon, in that if α is taken to be the confidence level, VaR then corresponds to the α quantile.[2] By convention, this worst loss is always expressed as a positive percentage in the manner indicated. In formal terms, then, if we take L to be the loss, measured as a positive number, and α to be the confidence level, then VaR can be defined as the smallest loss - in absolute value - such that:

$$P(L > VaR) \leq \alpha. \tag{1}$$

A more detailed definition of VaR can be find in Jorion (2007), Acerbi (2002) or Bertrand & Prigent (2012).[3]

The crucial step in the worldwide adoption of VaR was the Basel II Accord of 1999, which has resulted in nearly complete adoption of that measure (Basel III must be applied by 2019). More recently, Solvency II regulations (for insurers in Europe) have proposed VaR as a reference measure in determining required

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[1]Note that VaR does not give any information about the likely severity of loss by which its level will be exceeded.

[2]See Appendix A for the formal definition of a quantile function, the inverse of a distribution function.

[3]In terms of gains rather than losses, the VaR at confidence level α for a market rate of return X whose distribution function is denoted $F_X(x) \equiv P[X \leq x]$ and whose quantile at level α is denoted $q_\alpha(X)$ is:

$$-VaR_\alpha(X) = \sup\{x : F_X(x) \leq \alpha\} \equiv q_\alpha(X).$$

capital. The Basel Accord requires banks to recompute VaR periodically and to always maintain sufficient capital to cover those losses projected by VaR. Unfortunately, there exists more than one measure of VaR, since volatility, a fundamental component of VaR, remains latent. Therefore, banks must make use of several VaR models - at least for backtesting purposes - and so compute a range of prospective losses.

In this paper, we do not directly address VaR's appropriateness as a risk estimator, nor the adequacy of this measure for risk budgeting purposes. It suffices that regulators have seen fit to choose the VaR measure for required economic capital calculations, and that its computation is mandatory for all regulated practitioners. VaR is thus an essential research subject and of considerable interest to a broad spectrum of academics.

Methods to compute VaR or to determine distribution quantiles have already been the subject of considerable research, following VaR's introduction into current banking practice. We note some fundamental articles on VaR assessment and methods of its determination, among them Monte Carlo simulations: Pritsker (1997); Johnson transformations: Zangari (1996a); Cornish-Fisher expansions: Zangari (1996b) and Fallon (1996); the Solomon-Stephens approximation: Britten-Jones & Schaefer (1999); saddle-point approximations: Feuerverger & Wong (2000); Fourier-inversion: Frolov & Kitaev (1998); and extreme value theory: Longin (2000).

VaR has been the subject of numerous papers in real estate, but they have primarily focused on listed (i.e. securitized) real estate and not direct commercial real estate.[4] VaR for securitized real estate relies on the same methods as those used for ordinary stocks and bonds. Zhou & Anderson (2012) concentrate on extreme risks and the behavior of REITs in abnormal market conditions. They find that estimation of the risks requires different methods for different stocks and REITs. Cotter & Roll (2010) study REIT behavior over the past 40 years, highlighting the non-normality of REIT returns. They compute VaR (called risk of loss in their paper) for the Case & Shiller index, following three methods that do not rely on Gaussian assumptions, these being the Efficient Maximization algorithm, the Generalized Pareto Distribution method, and the GARCH model. Liow (2008) uses extreme value theory to assess VaR dynamics of ten major securitized real estate markets, allowing one to evaluate the risk of rare market events better than would be possible using traditional standard deviation measures.

Literature focusing on VaR in the context of direct real estate investment (or funds) is sparse. Nonetheless, some studies do concentrate on risk management and assessment in real estate. Booth et al. (2002) examine risk measurement and management of real estate portfolios, suggesting that practical issues force real estate investors to treat real estate differently from other asset classes. The report focuses on the difference between symmetric measures, such as standard deviation, and downside risk measures, such as VaR. Their work concentrates on all risk measures used in real estate, thus constituting a survey of then-

[4]For instance, the former encompasses REITs.

current real estate risk measures. Gordon & Tse (2003) consider VaR as a tool to measure leveraged risk in the case of a real estate portfolio. Debt in a real estate portfolio is a traditional issue much studied in real estate finance. Their paper demonstrates that VaR allows better assessment of such risk. In particular, traditional risk-adjusted measures (e.g., the Sharpe or Treynor ratio as well as Jensen's alpha) suffer from a leverage paradox. Leverage adds risk along with potential for higher returns per unit of greater risk. Therefore, the risk/return ratio does not change noticeably and so does not constitute an accurate tool by which to measure the risk inherent in debt. Contrarily, VaR is quite a good tool for studying leveraged risk. Brown & Young (2011) focus on a new way to measure real estate investment risk using spectral measures. They begin by refuting the assumption of normally distributed returns, whose adoption serves to flaw forecasts and decisions. Interestingly, VaR is not their selected measure; instead, an Expected Shortfall technique is adopted.

The limited research focusing on direct commercial real estate risk, in spite of the increasing interest in the topic, is likely due to both a lack of data from the commercial real estate sector and to issues arising from non-normality of returns. Limited data for this sector is one of the primary obstacles to reliable VaR computation. Either you invest in listed real estate and it is quoted daily with sufficient data available to compute VaR for your portfolio, or you invest in direct real estate and you deal with small data sets. This is particularly true in commercial real estate, in which investment is largely done by large institutions. The real estate market is thus comparable to the private equity market, where indices are created from small numbers of transactions. Any real estate property index attempts to aggregate real estate market information in order to provide a representation of underlying real estate performance. However, observation is generally conducted monthly in the best of cases, else quarterly, semi-annually, or sometimes even annually: it has largely to do with the sector under consideration. The residential field, where many transactions are observable, frequently features a monthly index. Commercial real estate (e.g., offices, shopping centers etc.) faces greater difficulty in regularly delivering data, and the indices are consequently of longer periodicity. To determine VaR of a real estate portfolio at threshold 0.5% (as requested by the Solvency II framework) using the historic approach, a minimum of 200 values are needed, which represents 17 years even for a monthly index. With that number of needed observations, VaR considerations are frequently irrelevant, since this requirement typically exceeds the recorded history of the index. Hence, it is necessary to use other methods in order to determine VaR for direct commercial real estate.

The non-normality of real estate return distributions is another perplexing issue for VaR computation. This point has been ably demonstrated by Myer & Webb (1994), Young & Graff (1995) and Byrne & Lee (1997). Recent studies such as Lizieri & Ward (2000), Young et al. (2006) and Young (2008) show that real estate returns usually exhibit non-normal returns. These works focus mainly on Anglo-Saxon economies, but similarities in real estate return distributions are found elsewhere. Real estate returns typically lean to the left (skewness) and exhibit fat tails (leptokurtosis). The distribution used to estimate VaR of a portfolio needs to be determined from such return distributions or corresponding sector indices. Nonetheless, an inappropriate normality assumption is regularly adopted in order to determine VaR, mainly because it allows quick and easy computation.

Both the lack of data and the non-normality issues which must be considered when determining commercial real estate VaR constitute the principal motivation for our study. The issue of Solvency II regulation (European controls on insurers) is particularly interesting. These regulations base capital requirements on VaR estimation, and propose a standardized approach. Nonetheless, these authorities leave open the possibility of building further in-house ("internal") models. Application of their standard model to real estate VaR estimation leads to a required capital of 25% for typical real estate investments in Europe. This calculation, however, was made using the IPD UK Monthly Property Index Total return, which then only applies to the UK, even though the resulting regulations concern all of Europe. The reasons are both that this index is published monthly (which we saw is frequent in real estate) and that this index is one of the few reliable commercial indices available in Europe. The committee recognizes the non-normality of real estate returns, but nonetheless does not attempt to more reliably estimate required capital for real estate. This is in part because they identify a lack of data among other difficulties in computing VaR. While the regulators recognize the liberties taken in their analysis, they make little effort to provide solutions or answers: "All distributions of property returns are characterized by long left fat-tails and excess kurtosis signifying disparity from normal distribution. [...], albeit the methods do not eliminate the inherent bias." The committee also recognizes its conservatism in using the total return index as a basis for calculation, since this inherently assumes that rental yield earned is re-invested at a similar rate. In sum, regulators admit the various inadequacies and imperfections in their recommendations, but in the absence of better data and because of the low proportion of real estate portfolios held by insurers in Europe, they do not attempt to improve on their preliminary work. The present effort is meant to be an advancement on this piece of regulatory analysis, though the approach taken here applies equally well to other regulations based on VaR (or quantile) estimation.

While the regulators suggest applying 25% VaR to all European countries, as historically computed using the U. monthly total return index, this conclusion, as they concede, is not much different than the one that would be obtained under an inappropriate Gaussian assumption. In view of this, we seek VaR methods that explicitly consider the non-normality of real estate returns in performing VaR computations, while still not relying on large data sets. This is indeed what the Cornish-Fisher expansion accomplishes. The expansion uses moments of orders higher than two, and thus deals with asymmetric, non-normal distributions. In short, the Cornish-Fisher approximation transforms naïve Gaussian quantiles according to the skewness and kurtosis coefficients taken to characterize the true distribution. Proper use of the Cornish-Fisher expansion needs only avoid one important pitfall: the formula is valid only if skewness and kurtosis coefficients of the risk distribution meet a particular constraint. In practice, this constraint is not usually treated, but here we employ a rearrangement procedure to remedy this problem, applying the technique in a commercial real estate context.

To the best of our knowledge, the use of Cornish-Fisher expansion to determine VaR in real estate has not been the subject of extensive research. No study focuses solely on Cornish-Fisher methods in a direct real estate context. Lee & Higgins (2009) do use Cornish-Fisher expansion in real estate, arguing that the Sharpe performance formula neglects two important characteristics of real estate returns: non-normality and autocorrelation. They then apply the Cornish-Fisher expansion to adjust the Sharpe ratio so as to account for this non-normality. Farrelly (2012) presents a study that focuses on measuring the risk of an unlisted property fund using a forward-looking approach. Among other relevant analysis, the author considers moment measurements of orders higher than two (so that asymmetry is treated) by using the Cornish-Fisher expansion. Following these authors, we motivate our paper by a need for better VaR assessments, but in the direct real estate field.

Studies examining the best methods to compute VaR in various other contexts and which then consider the Cornish-Fisher method among others do exist in greater numbers. Pichler & Selitsch (1999) compare five VaR methods in the context of portfolios and options, mainly: Johnson transformations, Variance-Covariance, and the three Cornish-Fisher-approximations of the second, fourth and sixth order, respectively. They conclude that a sixth-order Cornish-Fisher approximation is best among the approaches treated. Mina & Ulmer (1999) compare Johnson transformations, Fourier inversion, Cornish-Fisher approximations, and Monte Carlo simulation, concluding that Johnson transformations do not constitute a robust technique. Monte Carlo and Fourier inversion, instead, are robust, while the Cornish-Fisher approach, though fast, is a bit less robust, particularly when the distribution is far from normal, this being due to the aforementioned possible non-monotonicity of the Cornish-Fisher expansion. Feuerverger & Wong (2000) focus on when to use a Cornish-Fisher expansion in comparison to Fourier inversion, saddle point, and Monte Carlo approaches. They propose an extension of the Cornish-Fisher method which includes higher-order terms. Jaschke (2001) concentrates on properties of the Cornish-Fisher expansion and its underlying assumptions in the context of VaR, with particular focus on non-monotonicity of the distribution function, when convergence is then not guaranteed.[5] Jaschke discusses how the conditions for its applicability make the Cornish-Fisher approach difficult to use in practice. However, he demonstrates that when a data set obeys the required conditions, the accuracy of the Cornish-Fisher expansion is generally more than sufficient for one's needs, in addition to being faster than other approaches.

In summary, to estimate VaR of direct commercial real estate and unlisted property funds, we use a Cornish-Fisher expansion together with a rearrangement procedure ((Chernozhukov et al., 2010)). This method explicitly accounts for asymmetry and fat-tailed characteristics of direct real estate returns. We

[5] See also the chapter (by Jaschke and Jiang) of (Härdle, 2009) for a detailed presentation.

thus improve on both traditional models and that of regulators (the so-called "standard model"). Consequently, this paper contributes to extant literature by employing a method not based solely on the first two statistical moments, and which has particular pertinence given the current regulatory environment, i.e. recent regulations that base capital requirements on VaR.

The remainder of the paper is organized as follows. Section 2 introduces the Cornish-Fisher expansion and discusses technical points, while Section 3 implements the model and section 4 then concludes the paper.

2 Gaussian Value at Risk (VaR) and the Cornish-Fisher Adjustment

2.1 VaR with the Normality assumption

After a brief review of VaR in the Gaussian case, we analyze the Cornish-Fisher (CF) expansion and its implications for VaR computations. If the returns were normal, we would know the quantiles of the distribution exactly. Let X be a random variable that models returns distributed normally with mean μ and standard deviation σ, i.e. $X \sim N(\mu, \sigma^2)$. This random variable can be written as a function of the standard normal variable $z \sim N(0, 1)$ as follows: $X = \mu + Z\sigma$. We denote by z_α the standard Gaussian quantile at threshold α, which is to say, $F_z(z_\alpha) = \alpha$, where F is the cumulative distribution function (cdf) in the standard normal case. The quantile q_α for X is then identified by $F_X(q_\alpha) = \alpha$, for the corresponding distribution function F_X of the normal variable X and can then be written as $q_\alpha = \mu + z_\alpha\sigma$. In terms of returns X, one then has $VaR = -q_\alpha$, assuming an α such that q_α is negative.

2.2 VaR without the normality assumption: the Cornish-Fisher expansion

Cornish & Fisher (1938) established the expansion that bears their names. In the case of smooth random variables, it is possible to obtain an explicit expansion for any standardized quantile of the true distribution as a function of the corresponding quantile of the unit normal approximation introduced above. This CF expansion is then a simple polynomial function of the corresponding unit normal quantile, where the coefficients of each resulting term are functions of the moments of the true distribution under consideration.[6] For instance, denoting by z_α and $z_{CF,\alpha}$ the Gaussian and the resulting Cornish-Fisher quantiles, respectively we obtain the following expression for the normalized Cornish-Fisher quantile:[7]

$$z_{CF,\alpha} = z_\alpha + \frac{1}{6}(z_\alpha^2 - 1)S + \frac{1}{24}(z_\alpha^3 - 3z_\alpha)(K-3) - \frac{1}{36}(2z_\alpha^3 - 5z_\alpha)S^2, \forall \alpha \in (0, 1, \quad (2)$$

[6]This approximation is based on the Taylor series developed, for example, in Kendall et al. (1994) and Stuart & Ord (2009).

[7]At the third order, the approximation is: $\forall \alpha \in (0, 1), z_{CF,\alpha} = z_\alpha + \frac{1}{6}(z_\alpha^2 - 1)S$.

where S and K denote the skewness and kurtosis coefficients, respectively of the true distribution (see the definition of S and K in Appendix B).[8] The corresponding modified Cornish-Fisher quantile is then just:

$$q_{CF,\alpha} = \mu + z_{CF,\alpha}\sigma, \forall \alpha \in (0,1), \tag{3}$$

and so the expression for VaR is of course simply

$$VaR_{CF,\alpha} = -q_{CF,\alpha} \forall \alpha \text{ such that } q_{CF,\alpha} < 0. \tag{4}$$

The Cornish-Fisher expansion thus aims to approximate the quantile of a true distribution by using higher moments (skewness and kurtosis) of that distribution to adjust for the distribution's non-normality. Since the moments of the true distribution can be estimated in standard fashion by the sample skewness S and sample kurtosis K coming from data, these can then be substituted into equation (2) so as to estimate the unknown quantiles (the VaR) of the true distribution.

Cornish-Fisher expansion thus allows one to consider higher-order characteristics of the distribution when doing quantile computation, so that risky assets exhibiting non-normal distributions can be accurately treated. The Cornish-Fisher approach thus offers several advantages. First, it is comparatively easy to implement. Second, it allows for skewness and kurtosis in the VaR estimation, unlike the usual Gaussian approximation. Note that if the true distribution happened to indeed be normal, the Cornish-Fisher expression would simply reduce to the usual Gaussian expression. Third, the approach makes no assumption about the time scale and so can be repeated through time.[9] This renders the approach particularly relevant for, say, regulatory purposes. Indeed, the technique is independent of the nature of the underlying distribution and so of its evolution, and thus can be used whatever the changes in this distribution as the result of new, non-systematic events. This point is fundamental for risk management where, exactly as in accounting, one of the basic criteria is the "consistency principle", requiring that a company must be able to use the same risk measurements methods from period to period. Fourth, estimation using the Cornish-Fisher expansion does not require a large amount of data. For VaR computation, the relevant quantiles need to be estimated. With a sufficiently large data set, one could utilize a straightforward empirical quantile; however, in commercial real estate the available data is rarely sufficient for this task: the 0.5% VaR of Solvency II regulation requires at a very minimum 17 years of data (17 years = 204 months). If the return series is skewed and/or has abnormal tails (kurtosis), Cornish-Fisher estimates of VaR is the more appropriate than traditional methods since despite its need to determine skewness and kurtosis, the method only requires modest amounts of data.

As suggested, the Cornish-Fisher approach leads to approximations closer to the true law than does the traditional Gaussian approach, often to a dramatic

[8] Notice that in presence of a Gaussian distribution ($S = 0$ and $K = 3$), equation (2) reduces to the Gaussian quantile.

[9] However, exact distributions have advantages as well: they enable Monte Carlo simulations and so allow the direct computation of VaR.

degree. This is illustrated in figure 1 below for a chi-squared distribution with 4 degrees of freedom. Obviously, the CF methodology cannot give better results than does the true distribution, but it does have the benefit of approaching the true distribution far more closely than does the Gaussian approximation. By showing how close to the theoretical distribution the Cornish-Fisher can be, one begins to appreciate the power of this tool.

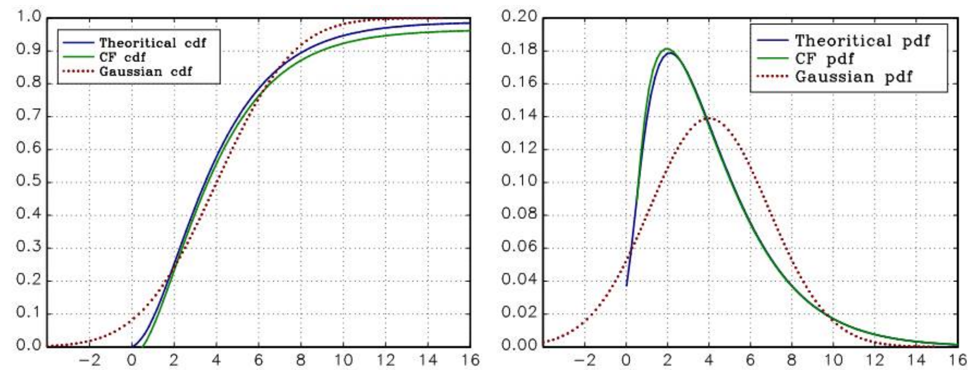

Figure 1: The Cornish-Fisher approximation for a chi-squared distribution with 4 degrees of freedom $\mu = 4$; $\sigma^2 = 8$; $S = 1.41$; $K = 6$

2.3 A Pitfall of the Cornish-Fisher Approach and its Solution

Although the Cornish-Fisher expansion has proven to be a useful technique, there are constraints on the permitted values of the true distributions' moments in order that the CF approximation itself yields a true distribution. Relation (2) in general allows a non-monotonic character to z_{CF}, which is to say that the true distribution's ordering of quantiles is not preserved. This violates a basic condition that must be met in order that the resulting CF approximation constitutes a proper cdf. Barton & Dennis (1952), Draper & Tierney (1973) and Spiring (2011), among others, study the domain of validity for the Cornish-Fisher expansion. Monotonicity requires the derivative of $z_{CF,\alpha}$ relative to z_α to be non-negative. This leads to the following constraint, which implicitly defines the domain of validity (D) of the Cornish-Fisher expansion:[10]

$$\frac{S^2}{9} - 4\left(\frac{K-3}{8} - \frac{S^2}{6}\right)\left(1 - \frac{K-3}{8} - \frac{5S^2}{36}\right) \leq 0. \tag{5}$$

Thus, if $(S, K) \in D$, then the CF quantile function is monotonic, but if it is not, the Cornish-Fisher method is inapplicable. Because of this, Chernozhukov

[10] For example, inequality (5) implies a kurtosis coefficient higher than 3 (a positive excess of kurtosis), which indicates a leptokurtic distribution. Thus, unadjusted CF expansion is not appropriate in the presence of thin tails.

et al. (2010) propose a procedure called increasing rearrangement in order to restore monotonicity. The procedure serves to sort the function of interest (see appendix D). As these authors demonstrate, the rearrangement procedure alters non-monotone approximations so that they become monotonic. In our problem, this corresponds to an ascending sorting of the quantile function $q_{CF,\alpha}$.

Consider some distribution with a skewness coefficient of 0.8 and a kurtosis of 2. These parameters correspond to a distribution being thin-tailed and right-skewed, respectively. Since these parameters do not belong to the domain of validity D, the $z_{CF,\alpha}$ quantile function is not monotonic. Applying the rearrangement procedure to this quantile function, though, we obtain $\tilde{z}_{CF,\alpha}$, the corrected Cornish-Fisher transformation of the Gaussian quantiles. Focusing on probabilities of less than 25%, figure 2 shows the impact of this procedure. Note that the computation of the non-rearranged CF probability density function is not always even possible since it would result in negative probabilities.

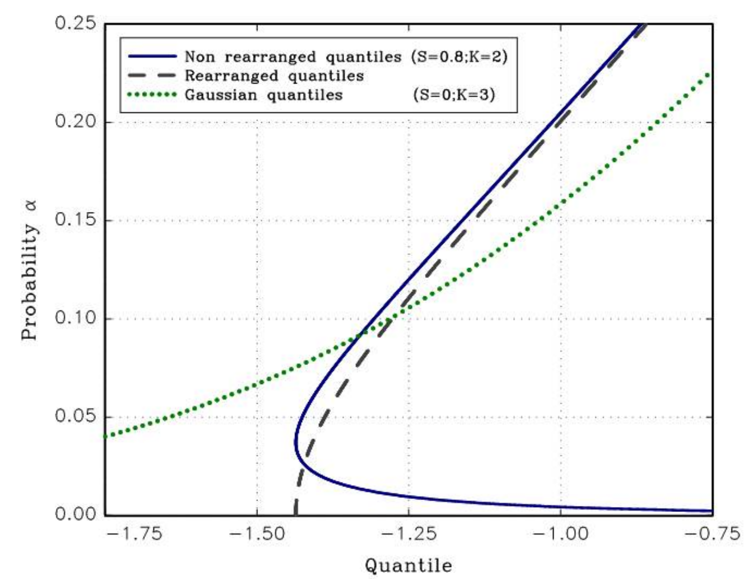

Figure 2: The rearrangement procedure ($\alpha < 25\%$)

Observe, also, that the discrepancy between the two quantile functions, the rearranged and the non-rearranged, is most noticeable for the smallest probabilities, which are the most important ones for VaR computation.[11]

The power of rearrangement is further illustrated, this time for a chi-squared distribution with 1 degree of freedom. In Figures 3 and 4 we can observe how the non-rearranged Cornish-Fisher quantiles function results in a rather poor

[11]See the first figure presented in Chernozhukov et al. (2010). Note that the non-rearranged quantile function encountered might be even more severely non-monotonic (and therefore could provide poorer approximations of the distribution function) than the one presented in figure 2. We note that $\tilde{z}_{CF,0.001} = -1.4$, whereas $z_{CF,0.001}$ is clearly biased, being equal to -0.3.

estimation of the ideal quantiles. On the contrary, the rearranged quantiles are very close to the theoretical ones, clearly showing how the rearrangement can improve the quality of the estimation. We thus see that the Cornish-Fisher approximation may provide a rather poor approximation of the quantile function when rearrangement procedures are not used, and that the rearranged approximation then provides a much more satisfactory approximation to the theoretical function than does the original approximation.

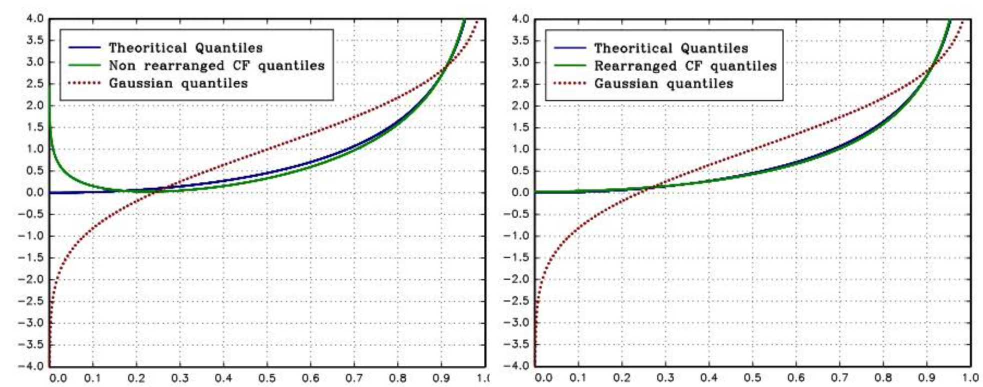

Figure 3: Comparison of the non-rearranged and rearranged Cornish-Fisher quantile functions for a chi-squared distribution with 1 degree of freedom $\mu = 1$; $\sigma^2 = 2$; $S = 2.82$; $K = 15$

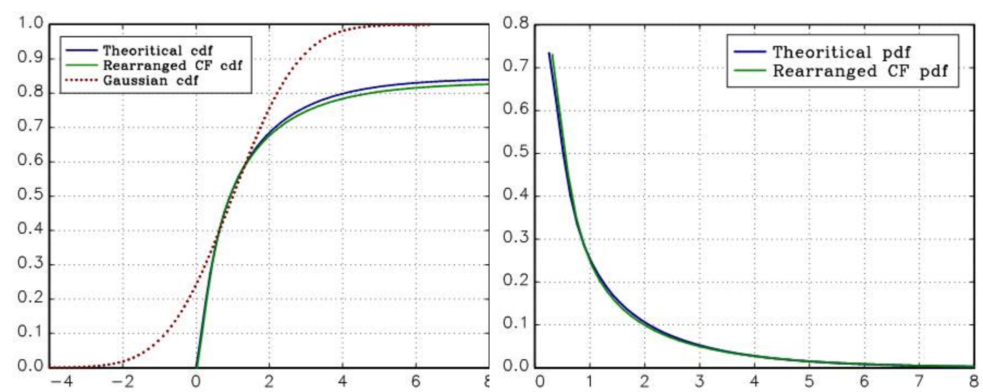

Figure 4: A cdf and pdf comparison of the rearranged Cornish-Fisher procedure for a chi-squared distribution with 1 degree of freedom $\mu = 1$; $\sigma^2 = 2$; $S = 2.82$; $K = 15$

The improvement in accuracy seen in the above example is not a fluke, since not only is rearrangement guaranteed to restore monotonicity of the approximation of the ideal distribution but it is also guaranteed to improve the accuracy of that approximation in comparison to the one achieved without rearrangement. The argument is presented in Chernozhukov et al. (2010). The improvement comes from the fact that, since the rearrangement results in monotonicity, it necessarily brings the originally non-monotone approximations closer to the true monotone target function.

3 A Commercial Real Estate Application

We study the IPD UK Monthly All Property Total Return Index from January 1988 to December 2010, which consists of 276 observations. By comparison, the regulations the European authorities developed for real estate (Solvency II) were based on the IPD UK Monthly Property Index Total return spanning 1987 to the end of 2008, totalling 259 monthly returns. This IPD index is a valuation-based index and is "like all indices" subject to criticism. Nevertheless, the limitations of this specific index - e.g., its smoothness and reliability - are not further discussed here, this not being the subject of our paper. Note that contrary to the work of regulators in the context of Solvency, we apply desmoothing procedures to our database. The desmoothing procedure[12] we have used is the one introduced by Geltner (1993), where we use a lag parameter of 0.5, corresponding to a lag of two months.[13] Our approach remains applicable, though, to most any kind of index, assuming the first four moments can be estimated, with our objective in this part merely being to impose our methodology on some commonly accepted and well-understood index. In this sense, the IPD UK Monthly All Property Total Return Index presents four advantages: monthly publication, reliability, acceptance by practitioners, and substantial representation of its components in institutional investors' portfolios. Our objective in the following, then, is to apply the Cornish-Fisher expansion in order to compute the VaR of UK real estate total returns. We apply our described techniques to this database and determine VaR at a 0.5% threshold, this threshold being, for instance, the one required by solvency II regulations. The values calculated are annualized monthly returns.

The index and corresponding returns on which the analysis is based are presented in figures 5 and 6, clearly exhibiting both the 1990's overconstruction crisis and the subsequent subprime crisis.

[12]An extensive presentation of the desmoothing technique can be found in Geltner et al. (2007) p.682. However, we note that the impact of the desmoothing is negligible in our case because of offsetting impacts on each of the first four moments

[13]Index issues have already been discussed in the Solvency II calibration paper CEIOPS-SEC-40-10, as well as in many articles; among others: Fisher et al. (1994), Edelstein & Quan (2006), Booth et al. (2002) and Cho et al. (2003).

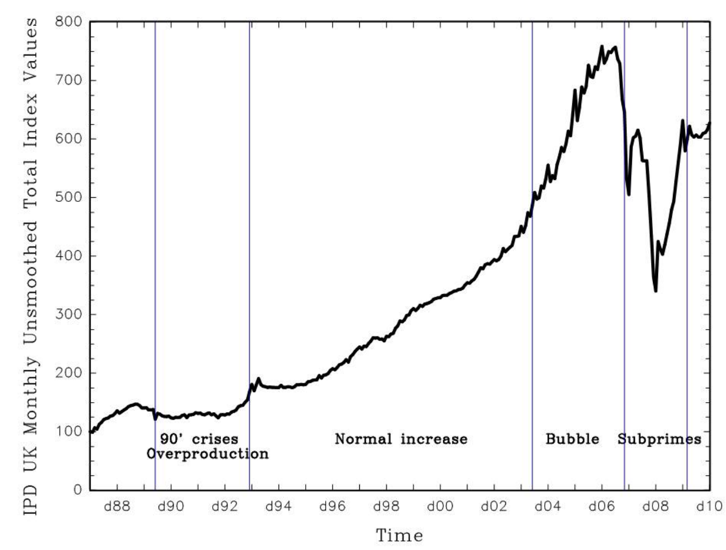

Figure 5: The real estate index from January 1988 to December 2010

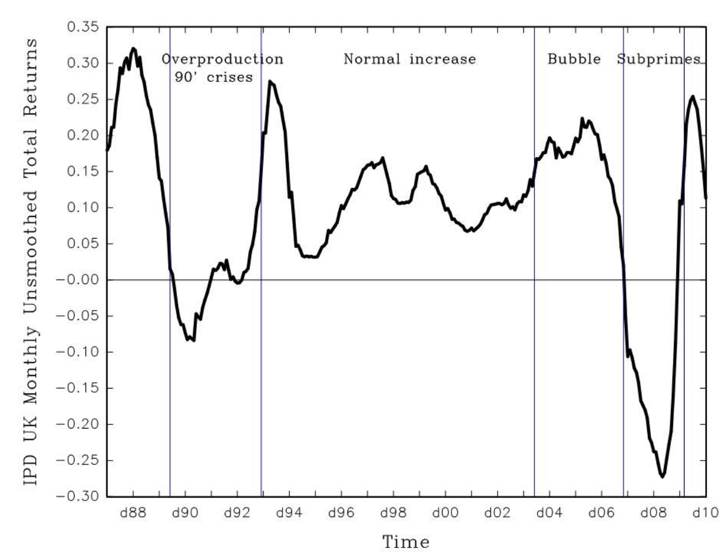

Figure 6: Real estate returns from January 1988 to December 2010

In order to determine VaR evolution, we need to choose an appropriate bandwidth. This choice of bandwidth is not a priori determined, either statistically or from a real estate viewpoint (e.g., by economic analysis, regulations, etc.). The length of time does have to be large enough to enable moment computation. Standard valuation models use a 10-year cash flow period, but the chosen length could be longer. For example, the Solvency II standard model for real estate required capital is based on all the observations then available. Since the purpose of our analysis is VaR for non-Gaussian distributions and not the impact of window length, we do not study this point further here (but see appendix E). A 15-years period is used, being a compromise between the need for sufficient data and the desire to exhibit the evolution of VaR over time. While still allowing us to consider more than one business cycle, this choice enables us to obtain results that are not overly erratic, as would be the case with smaller windows.

Distributions of returns differ across periods, as illustrated in figure 7. In addition to a curve corresponding to the entire period, there are also curves corresponding to three overlapping 15-years periods: the first 15-years period (January 1988 to December 2002), the middle 15 years (December 1991 to November 2006) and the last 15 years (December 1995 to December 2010).

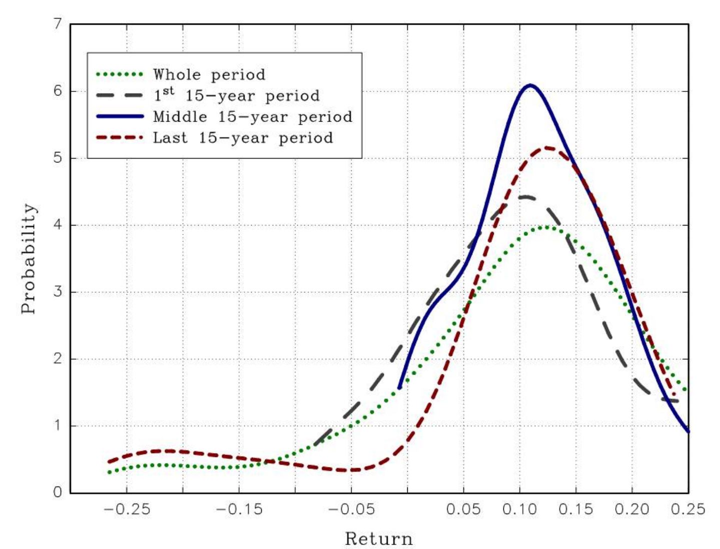

Figure 7: Real estate return probability distribution functions according to 15-years periods

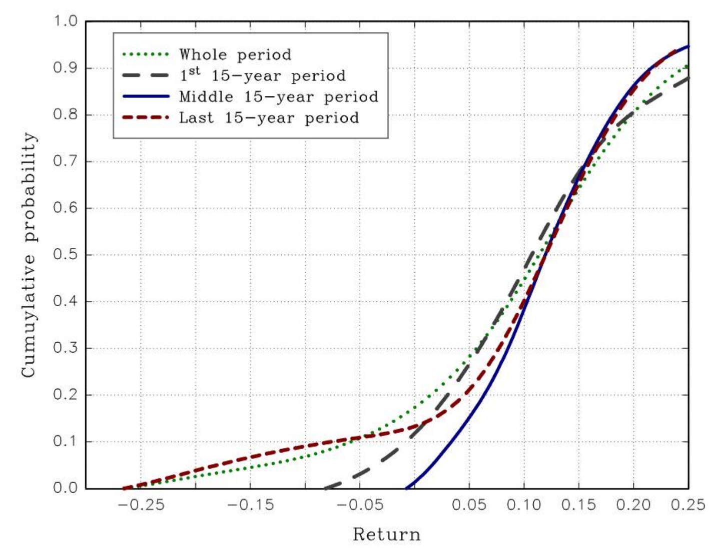

Figure 8: Real estate return cumulative distribution functions according to 15-years periods

The middle 15-years period distribution is the most concentrated one, with the high returns from the end of the 1980s and the lower returns from the subprime crises not being incorporated. The distribution for the last 15-years period is left-skewed, with highly negative returns (reflecting the subprime crisis).[14] These more recent events demonstrate how the naïve use of the Gaussian model becomes completely inappropriate at crucial times. Accompanying descriptive statistics of the empirical moments are presented in table 1.

Periods	Mean	S.D.	S	K	Q1	Q2	Q3
Whole period							
01/88-12/10	.0938	.1555	(.4483)	3.8172	.0301	.1074	.1814
1st 15-years							
01/88-12/02	.1040	.1198	.3909	3.4127	.0347	.0935	.1654
Middle 15-years							
12/91-11/06	.1234	.0843	.4991	4.5293	.0692	.1183	.1763
Last 15-years							
01/96-12/10	.0923	.1537	(.9720)	4.8255	.0677	.1134	.1756

Table 1: Basic statistics of the database for various periods

Figure 9 and 10 show the 95% bootstrap confidence interval of the mean, the standard deviation, the skewness coefficients, and the kurtosis coefficients

[14] This raises a questions concerning the window length of the data set chosen by regulators. A shorter window length leads to higher skewness and kurtosis coefficients nearer the present, since the subprime and mortgage crisis then have more weight.

through time. In both graphs, the values represent computations based on windows spanning the previous fifteen years periods. The means and standard deviations are seen to be rather unstable. Mean returns are observed to exhibit a W-type shape, with a small decrease till 2004, a solid increase up to 2007, followed by a fall until 2009, and then a slow recovery. The evolution of the standard deviation is nearly stable - though ever so slightly diminishing - before the crisis in 2007. Afterwards, it rises strongly.

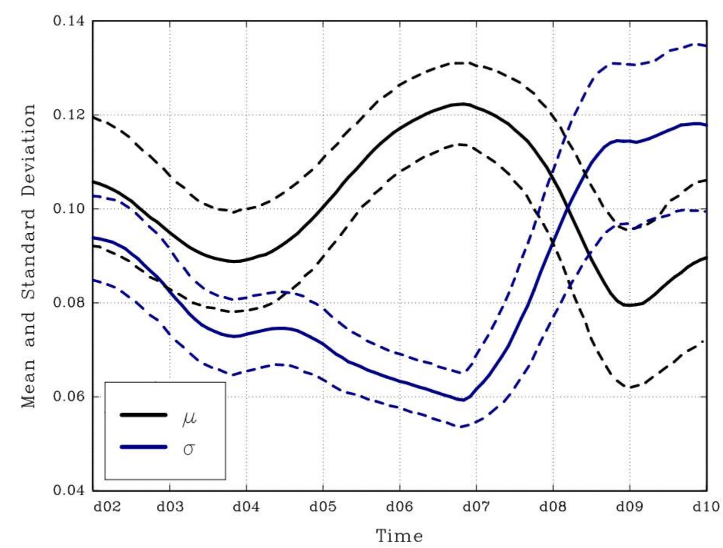

Figure 9: Mean and standard deviations according to 15-years periods, and their 95% bootstrap confidence intervals

In comparison, the evolution of S and K are a bit more variable. They are nearly stable around the Gaussian values of 0 and 3, respectively, before the subprime crises. However, changes are far more noticeable afterwards: the distribution becomes highly left-skewed ($S < 0$) and fat-tailed ($K > 3$). To the extent that it was not Gaussian, the distribution before the crisis appears to have been slightly thin-tailed, but after 2008 it becomes dramatically so. At about the same time, the kurtosis coefficient explodes in size.

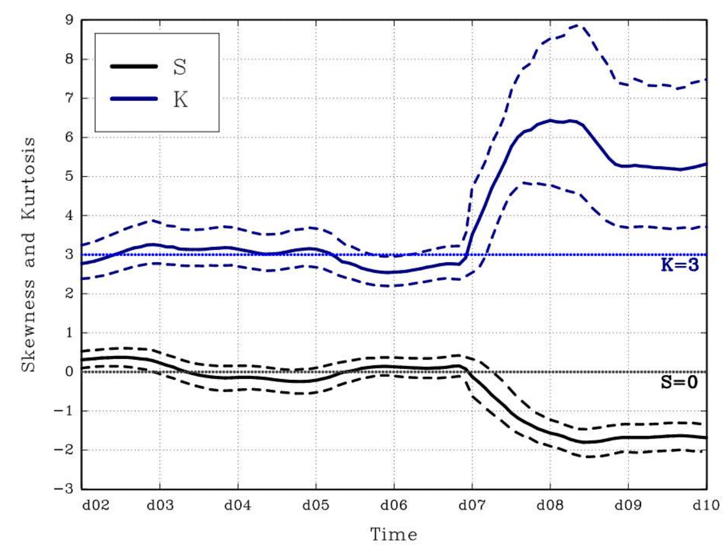

Figure 10: Skewness and kurtosis according to 15-years periods, and their 95% bootstrap confidence intervals

As mentioned in Section 2 of this paper, use of the Cornish-Fisher expansion is subject to a monotonicity condition. The latter is satisfied when skewness and kurtosis coefficients belong to the domain of validity D, defined by inequality (5). In particular, this condition is not satisfied any time kurtosis is lower than 3. The combination of skewness and kurtosis coefficients found in the data leads to a considerable period of invalidity. Without the rearrangement technique, the Cornish-Fisher expansion would be possible only in two relatively small periods: from November 2003 to April 2004 and from March 2008 to November 2008 (shown in figure 11).

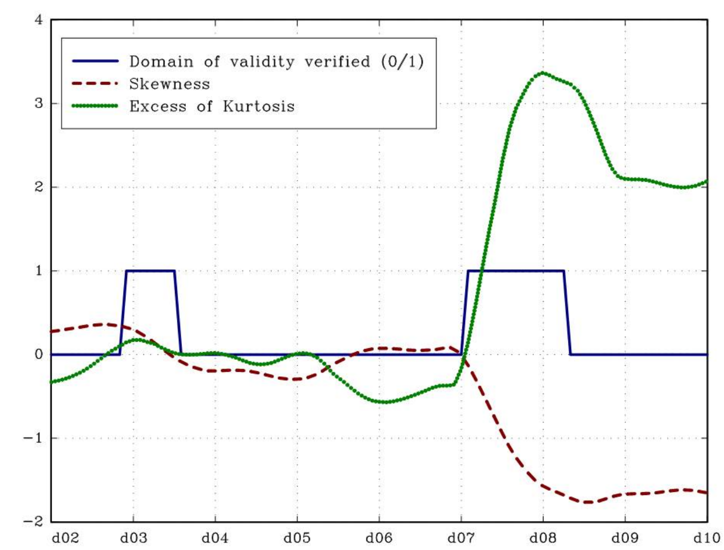

Figure 11: Skewness and kurtosis belonging to the domain of validity (D). The function represented is set to 1 if $(S, K) \in D$ and to 0 if $(S, K) \notin D$

We compute the probability distribution function from the empirical data for the first 15-years in figure 12 and for the last 15-years in figure 13 showing in the same figure the rearranged Cornish-Fisher result and the Gaussian result over the sample period.[15] This allows us to demonstrate the power and the relevancy of our proposed approach: clearly the Cornish-Fisher approach is closer to the empirical law than is the Gaussian one. This comes from the improvement resulting from the rearrangement: this technique brings non-monotone approximations closer to the monotone target functions, as logically demonstrated in Chernozhukov et al. (2010) (see appendix D).

[15]The halt in estimation is due to the lack of data required for kernel estimation.

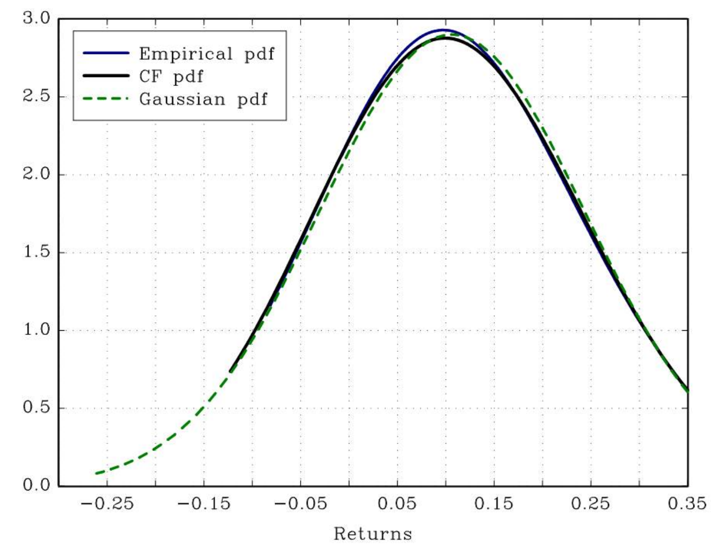

Figure 12: Probability distribution function of the 1st 15-years for the empirical, the Cornish-Fisher, and the Gaussian distributions

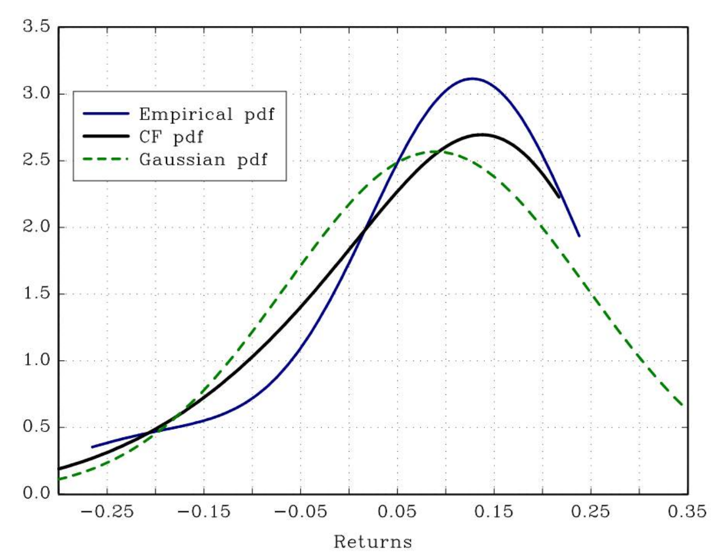

Figure 13: Probability distribution function of the last 15-years for the empirical, the Cornish-Fisher, and the Gaussian distributions

Given S and K for each 15-years rolling period, we next compute the Cornish-Fisher correction of Gaussian quantiles in order to obtain VaR. Figure

14 presents the results for the 0.5% thresholds, with their associated confidence intervals.[16] Figure 14 highlights how the corrected VaR (the Cornish-Fisher VaR is denoted $CF\ VaR$ in the graph) is different from the Gaussian one. During the rise of the bubble in the 2000's and then the subsequent collapse, the difference between the two VaRs becomes quite noticeable: the corrected one being lower before the crisis but much higher afterwards. This underlines one of the advantages of the current approach: in periods of low risk, the VaR obtained is lower than the Gaussian VaR, while to the contrary, when risk increases, then so does VaR. This is completely in line with the idea that required capital should be higher for riskier investments. In this sense, when real estate risk increases (due to overproduction, strong increase of price, abnormal transactions, etc.), the required capital ought to increase at the same time. This is exactly what $CF\ VaR$ accomplishes.

Another interesting point is to compare our results with those obtained by regulators. Solvency II regulation requires required capital of 25% for real estate investments. This value is based on the IPD UK Monthly Property Index Total return over the whole available period with a threshold of 0.5%. Interestingly, the valuation by regulators is close to that obtained with the Gaussian assumption at the same 0.5% threshold on the most recent 15-years rolling period: 22%. However, this is not the case for the $CF\ VaR$, which is much higher than the Gaussian one, being around 31%, an increase of 24% compared to the regulator's calculus. This observation raises questions about the relevancy of the regulator's VaR estimation. Two suppositions can be entertained: either the regulators do not care about correctly assessing real estate risk given the low proportion held by investors or the regulators do not really understand the specificities of real estate. In any case, the regulator's result is in favour of the real estate industry (because it undervalues the real risk) but to the disadvantage of real estate risk management. Clearly, including moments of orders higher than two leads to a more accurate VaR and subsequently to more accurate capital requirements (be they lower or higher).

[16] There are two sources of risk in the confidence interval computation for Gaussian VaR: randomness of mean and randomness of variance. Adding to these the randomness in the skewness and in the kurtosis coefficients, we obtain four elements entering the calculus of the corrected VaR. However, the 95% confidence interval of the corrected VaR is nonetheless observed to be smaller than the Gaussian one. There are more sources of randomness, but since these random variables correlate, the standard deviation of the VaR estimator is lower all the same.

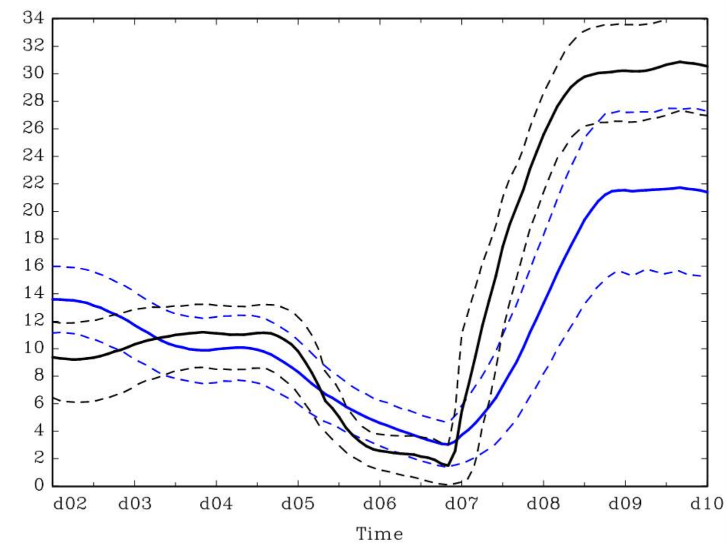

Figure 14: 0.5% Gaussian and corrected Cornish-Fisher VaR in percentage according to 15-years periods, with their corresponding 95% bootstrap confidence intervals

4 Conclusion

The challenge of risk modelling is to adequately incorporate the distribution of returns, since the under or overestimation of risk can alternatively lead to high losses or to significant missed opportunities. The aim of this paper has been to compute Value at Risk (VaR) for direct commercial real estate investment. The existence of substantial skewness and kurtosis in monthly commercial real estate returns, combined with the lack of data that the real estate industry faces, results in a systematic mis-estimation of risk when using the conventional VaR computation approach.

A key feature of our analysis has thus been to deal with the difficulties for standard risk modelling posed by the specificities of commercial real estate markets. In light of all the recent (Solvency, Basel) regulation that has followed the subprime crisis, risk measurements - in particular VaR estimates - are in great demand by the real estate industry, as well as by regulation authorities. Yet, to date, little research has concentrated on VaR analysis - or more generally on risk measurement - in the case of direct commercial real estate. This paper fills this lacuna by employing an approach based on Cornish-Fisher expansions - thus relying on higher order moments of returns - which results in an overall improvement in real estate VaR computation, since the resulting technique proves sensitive to the characteristics of the underlying true return distribution. Our article thus contributes to the extant literature by proposing a new approach to commercial real estate risk assessment.

One possible weakness of the Cornish-Fisher approach is in its domain of validity. This condition can be quite restrictive and may lead to wrong quantile estimations when the required criteria is not fulfilled. In this case, the monotonicity of the quantile function is not maintained, a mandatory condition for any sort of cumulative distribution function computation. This limitation on Cornish-Fisher expansions can be removed, though, by the use of rearrangement procedures (sorting schemes) for the quantile function, thus resolving the non-monotonicity issue. Here, we propose a methodology that combines Cornish-Fisher expansion with a rearrangement procedure in order to accurately compute real estate VaR.

The article applies the proposed methodology to a UK commercial real estate database (IPD UK Monthly All Property Total Return Index). We demonstrate that this index provides an example where Cornish Fisher VaR cannot be properly computed without the use of a rearrangement procedure. In examining the evolution of the first four moments of the index's distribution, we find that the skewness and the kurtosis move strongly through time, especially during and after the market collapsed. We are able to compare our results with those of the European Solvency II (25%) regulatory analysis. As anticipated, we obtain a higher VaR with the Cornish Fisher calculations than those of the regulators, except in times when the markets were especially calm. Indeed, while calculation shows that in recent periods our Gaussian VaR remains close to the 25% value obtained overall in the regulatory analysis, the corrected Cornish Fisher VaR is more like 31%. We thus see that ignoring higher moment corrections to VaR calculations can be of considerable consequence.

The Cornish-Fisher approach does not depend on any distributional assumptions and so may be the preferred choice when the distributional assumptions required by other modelling approaches are likely to be violated, e.g. when the return series does not follow the normal distribution assumed by numerous formulations. Similarly, using our methods, we can obtain meaningful results despite a relative paucity of data, which would render many other approaches completely inapplicable. These advantages may argue for using our approach in a more general risk management and assessment context. Hence, there are good reasons for real estate practitioners, as well as banks and insurers, to implement this method alongside other models when working in a real estate context, or whenever data sets prove modest. The proposed approach can additionally be used for regulatory purposes as a proxy for true VaR when conducting control and backtesting procedures.

Finally, while our paper has limited the use of its techniques to the IPD UK Monthly All Property Total Return Index for the class of public commercial real estate assets found in the U.K., individuals with portfolios invested in numerous other asset classes - such as private equity, arts, or hedge funds - which exhibit common features to commercial real estate assets (i.e, little data, abnormal distributions, etc.) may also profit from the use of our approach. It should be possible and potentially quite interesting to apply our model to risk comparisons among these various asset classes and then to apply this to optimal portfolio choice. Risk managers with a need to develop appropriate models of risk should find a useful approach here, one yielding "internal models" applicable to real estate, as well as to many other areas.

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A Appendix 1: Quantile Functions

The quantile function (or inverse cumulative distribution function) of the probability distribution of a random variable specifies, for a given probability, the value which the random variable will fall below with that specified probability. In fact it is an alternative to the probability density function (pdf).

Let X be a random variable with a distribution function F, and let $\alpha \in (0, 1)$. A value of x such that $F(x) = P(X \leq x) = \alpha$ is called a quantile of order α for the distribution. Then we can define the quantile function by:

$$q_\alpha(X) \equiv F^{-1}(\alpha) = inf\left\{x \in \mathbb{R} : F(x) \geq \alpha\right\}, \alpha \in (0, 1).$$

The quantile function $q_\alpha(X)$ thus yields the value which the random variable of the given distribution will fail to exceed with probability α.

B Appendix 2: Skewness and Kurtosis

Given a probability distribution $f(x)$ of the random variable X and a real-valued function $g(x)$, one defines the expectation $E[g(X)] = \int g(x)f(x)dx$, in which case the first moment is $\mu = E[X]$, whereas the higher central moments are then defined as $\mu_n = E[(X - \mu)^n]$. The first task in almost all statistical analyses is to characterize the location and variability of a data set. This is captured by the moments of order one and two, usually called the mean μ and the variance $\sigma^2 = \mu_2$, respectively. A further characterization of the data often includes the standardized moments of order three and four, called the skewness $\gamma_1 = \mu_3/\sigma^3$ and kurtosis $\beta_2 = \mu_4/\sigma^4$, respectively. These last two measurements further describe the shape of a probability distribution. We briefly recall the significance of these two last parameters.

Skewness is a measure of symmetry, or more precisely, the lack of symmetry. A distribution, or data set, is symmetric if it looks the same to the right and left of its center (which is then the mean μ). The skewness of any symmetric distribution, such as a Gaussian one, is necessarily zero. Negative values for the skewness coefficient indicate data that are skewed to the left whereas positive values indicate that the data that are right skewed, with left-skewness meaning that the left tail of the distribution is long relative to the right one.

Kurtosis refers to whether the data are peaked or flat relative to a normal distribution. That is, data sets with high kurtosis tend to have a distinct peak near the mean, then decline rather rapidly, but still have heavy tails. Data sets with low kurtosis tend to have a flat top near the mean rather than a sharp peak. The kurtosis formula measures the degree of this peakedness, where the kurtosis of a Gaussian distribution turns out to be 3.

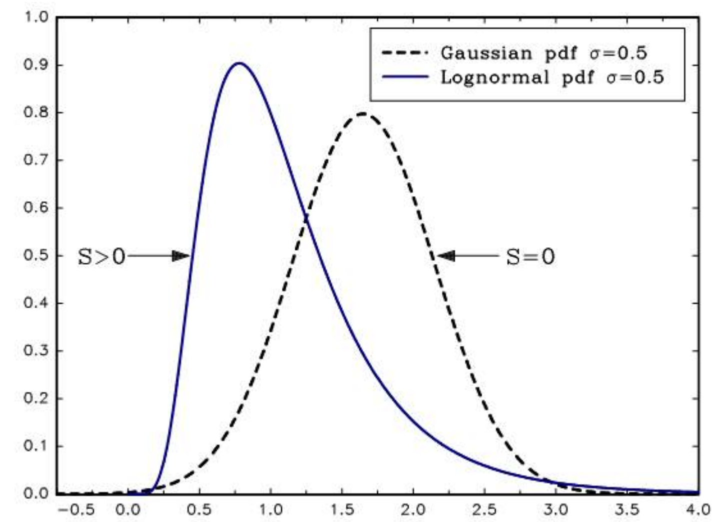

Figure 15: Right skewed distribution ($S = 1.75$)

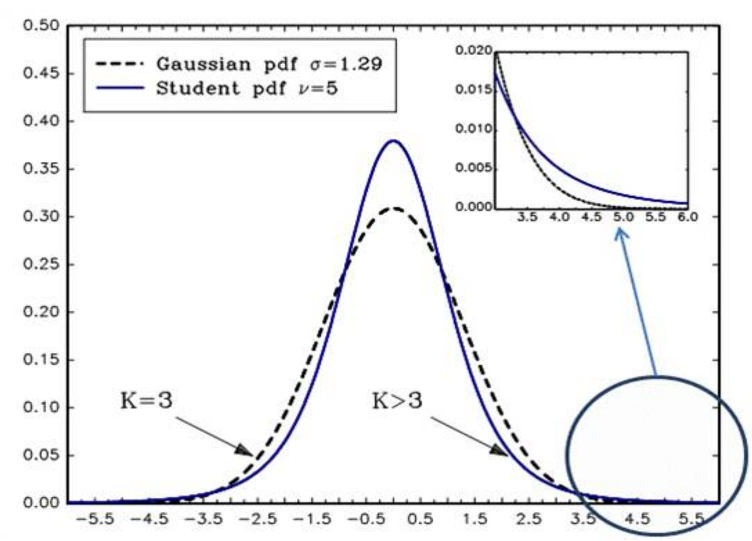

Figure 16: Fat tailed distribution ($K = 9$)

C Appendix 3: The Cornish-Fisher Procedure

The Cornish-Fisher expansion is a useful tool for quantile estimation. For any $\alpha \in (0,1)$, the upper αth-quantile of F_n is defined by $q_n(\alpha) = inf\,\{x : F_n(x) \geq \alpha\}$, where F_n denotes the cdf of $\xi_n = (\sqrt{n}/\sigma)(\bar{X} - \mu)$ and $\bar{X}$ is the sample mean of i.i.d. observations $X_1, \ldots, X_n$. If z_α denotes the upper αth-quantile of $N(0,1)$, then, the fourth order Cornish-Fisher expansion can be expressed as:

$$q_n(\alpha) = z_\alpha + \frac{1}{6\sqrt{n}}(z_\alpha^2 - 1)S + \frac{1}{24n}(z_\alpha^3 - 3z_\alpha)(K-3) - \frac{1}{36n}(2z_\alpha^3 - 5z_\alpha)S^2 + o(n^{3/2}),$$

where S and K are the skewness and kurtosis of the observations X_i.

The Cornish-Fisher expansion is useful because it allows one to obtain more accurate results than using the central limit theorem (CLT) approximation, which would be just the z_α defined in the body. A demonstration and example of the greater accuracy that the Cornish-Fisher expansion brings compared to the CLT approximation is reported in Barndorff-Nielsen & Cox (1989), p. 119.

D Appendix 4: The Rearrangement Procedure

This paper applies a procedure called rearrangement, or more precisely, increasing rearrangement. We use this procedure to restore the monotonicity of the Cornish-Fisher expansions. The procedure is briefly described here.[17]

A convenient way to think of the rearrangement is as a sorting operation: given values of a data set, we simply sort the values in an increasing order. The function created is the rearranged function.

Following Chernozhukov et al. (2010), we define the procedure more precisely as follows. “Let χ be a compact interval. Without loss of generality, it is convenient to take this interval to be $X = [0,1]$. Let $f(x)$ be a measurable function mapping χ to K, a bounded subset of $\mathbb{R}$. Let $F_f(x) = \int_\chi 1\{f(u) \leq y\}\, du$ denote the distribution of $f(x)$ when X follows the uniform distribution on $[0,1]$. Let

$$f^*(x) = Q_f(x) = inf\,\{y \in \mathbb{R} : F_f(y) \geq x\}$$

be the quantile function of $F_f(y)$. Thus,

$$f^*(x) = inf\left\{y \in \mathbb{R} : \left[\int_\chi 1\{f(u) \leq y\}\, du\right] \geq x\right\}.$$

This function f^* is called the increasing rearrangement of the function f.”

In our approach, this allows us to respect one of the basic conditions of the probability distribution function: monotonicity. As a result, Value at Risk becomes inversely proportional to the threshold, and so, as expected, one has $VaR_{0.5\%} \geq VaR_{5\%}$.

The rearrangement procedure also has the practical implication, demonstrated by Chernozhukov et al. (2010), that the resulting rearranged estimate has a smaller estimation error (in the Lebesgue norm) than does the original estimate whenever the latter is not monotone. This property is independent of the sample size and of the way the original approximation is obtained. Thus,

[17] In mathematics, the notion of rearrangement derives from the notion of permutation and is reported in the work of Bóna (2004). Lorentz (1953) can also be consulted.

the benefits of using a rearrangement procedure in our paper are both to obtain estimates of the distribution satisfying the logically necessary monotonicity restriction and also to obtain better approximation properties.

E Appendix 5: Comparison of Cornish-Fisher VaR for Various Window Lengths

Here, we revisit the length of window choice. In figure 17 and 18, VaR of the 10, 11, 12.5, 14 and 15-years periods are represented simultaneously. Before the middle of 2008, the longer the window, the higher is the VaR. After that date, we observe the opposite effect. Modifications created by the window length are qualitatively the same for the Gaussian and Cornish-Fisher VaRs.

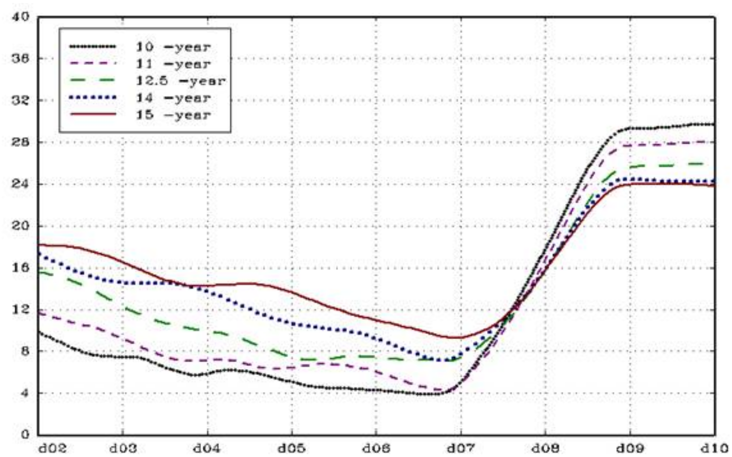

Figure 17: Gaussian VaR for various window lengths

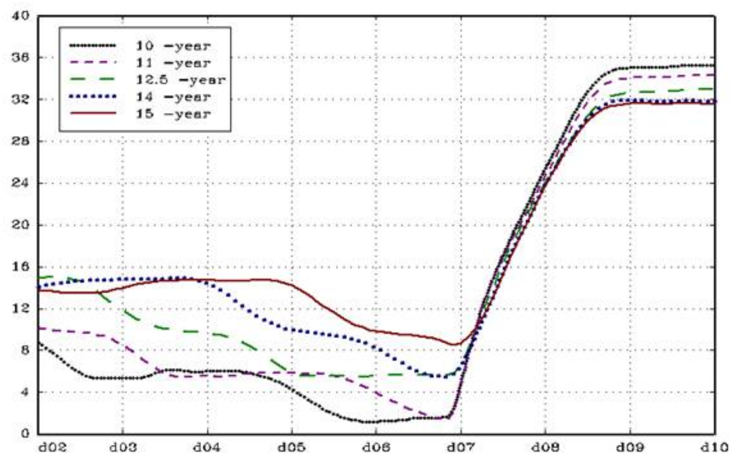

Figure 18: Cornish-Fisher corrected VaR for various window lengths

The difference between these two approaches to VaR is stable whatever the choice of windows. This illustrates that effects of window length is not relative to the Cornish-Fisher expansion but to the VaR computation, and more generally to distribution estimation. As mentioned previously, regulators often fix the length exogenously, with a 10-year window being chosen in much financial analysis.