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Working Paper CEMOTEV n° 02-2018

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Abstract

Standard results about portfolio optimization suggest that the allocation to real estate in a mixed-asset portfolio should be around $15-20\%$. However, the institutional investors share in real estate is significantly smaller, around $7-9\%$. Many researches have addressed this point even if as of today no consensus has emerged. In this paper, we built-up an allocation model that can explain the empirical observed weights. For this purpose, we account for the term structure of all standard financial assets and also of real estate asset class (expected returns, volatilities and correlations depending on the time to maturity). We propose a dynamic portfolio optimization model that allows analyzing portfolio weights with respect to the whole term structure modelling, due to its tractability and its good fit when being adequately calibrated. In this framework, we provide explicit and operational solutions to the dynamic mixed-asset portfolio allocation (cash, real estate, stock and bond). The results show that accounting for investment horizon and mean-reverting dynamics allows to better examine how portfolio allocations depend on both risk aversion and investment horizon.

Keywords: Portfolio allocation; Mixed-asset; Real estate investment; Mean reverting effects.

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1 Introduction

Mixed asset portfolio allocation consists usually in determining optimal weights invested on cash, bonds, stocks and finally real estate assets. For both institutional investors and academic researchers, it has been a very important topic from many years (see e.g. Webb *et al.*, 1986, 1987; Gold, 1986) and is still a rather involved problem. Indeed, the determination of the optimal allocation on real estate asset does not rise to a clear consensus, even almost all investors agree that real estate can be an effective portfolio diversifier (see e.g. Hoesli *et al.*, 2004). Hoesli and MacGregor (2000) suggest that allocation on real estate would be equal to about 15-20%. However, for instance, Fogler (1984) wonders whether one can justify a 20% weight invested on real estate. Hudson-Wilson *et al.* (2005) also deal with the rationality of real estate investment. Indeed, in many countries, looking at the portfolios of institutional investors, the weight invested on real estate is significantly smaller than 15-20%, as emphasized by Clayton (2007) and J.P. Morgan (2007) who show that the allocation on real estate is about 7.3% for the US and about 8.5% for the UK. Andonov *et al.* (2013) also find that the allocations on stocks and bonds dominate the others (about 75%) but the investment on alternative assets including real estate has increased over time.[1]

Summarizing results in the literature about mixed-asset investment, Seiler *et al.* (1999) show that real estate diversification ranges from 0% to 67%. Considering both public and private real estate in the mixed-asset framework (depending on the direct real estate index chosen), Feldman (2003) also finds that global real estate allocation ranges from 0% to 42%. Looking at the European financial market, Fugazza *et al.* (2007) determine the optimal weights for risk-averse investors, finding that the introduction of real estate assets in optimal portfolios yields to weights lying between 12% and 44%. They emphasize that "the welfare costs of either ignoring predictability or restricting portfolio choices to traditional financial assets only are found to be in the order of 150-300 basis points per year". According to Rehring (2012), the difference between theoretical weights and the low allocations to real estate in portfolios of institutional investors is viewed as a puzzle in real estate research (see Chun *et al.*, 2004). Lin and Liu (2008) point out also the heterogeneity of investors facing real estate returns and risk.As regards portfolio optimization, many empirical studies have

demonstrated how allocations between stocks, bonds and cash depend highly on risk aversion. In particular, bond/stock ratios differ for conservative, moderate or aggressive investors (see e.g. Brennan and Xia, 2000, de Palma and Prigent, 2009). Approaches to the portfolio allocation problem are usually based on the standard model of Markowitz (1952) where the usual portfolio maturities

[1] Andonov *et al.* (2013) examine about 880 pension funds in the United States, Canada, Europe, and Australia/New Zealand from 1990 to 2009. In 2009, the shares of these pension funds are respectively equal to 47.1% on stocks, to 36.9% on bonds, and to 2.5% on cash while the remaining amount (i.e. 13.5%) has been invested in alternative assets, especially 5.1% on Real Estate.

are monthly, quarterly or annual returns. Under specific assumptions (markets without friction, independent and identically distributed (i.i.d.) asset returns[2] and power utility to describe investor's utility), the optimal portfolio allocation is myopic. It means that, whatever the investment horizon, the weights are constant, implying that short-term and long-term asset allocations are equal. However, it is usually recognized that investors with longer investment horizon invest a higher percentage of their portfolio in stocks, which is not consistent with portfolio allocations determined from basic models of portfolio optimization (see e.g. Canner *et al.*, 1997). As emphasized by Bajeux-Besnainou *et al.* (2001), popular investment advice does not conform to the myopic property.

Additionally, the optimal portfolio allocation may depend significantly on the predictability of asset returns through their autocorrelations as emphasized for instance by Campbell and Viceira (2002). However the prediction of asset returns is difficult, especially for long time horizons. For instance the riskiness of stocks returns versus those of bonds may be questioning. For the real estate market, MacKinnon and Al Zaman (2009) examine the predictability of real estate returns in the U.S. They show that direct real estate returns exhibit mean reversion process and highlight how real estate investment risk is weaker for long-term investors. Introducing investment horizons in a buy and hold strategy as in Campbell and Viceira (2005), they conclude that real estate allocation increases in the long term due to the decline of long term correlations:[3] "real estate is a better diversifier for long-term portfolios than for short-term". In particular, they find (and use) a decreasing volatility over time and suggest an optimal allocation to real estate between 31% (versus 20% for short-term horizon, e.g. 1-year). Pagliari (2017) also shows how varying correlations between different asset classes modify the optimal mixed allocation. Adding the transaction costs and the marketing period risk to the return predictability, [4] Rehring (2012) finds the same conclusions for the U.K. commercial real estate: the allocation to real estate increases strongly with the investment horizon. The importance of return predictability for long-term horizons and of transaction costs for short and medium term is particularly highlighted. The author also finds a decreasing volatility of real estate returns over time.[5] The allocation to real estate is found to increase substantially with the investment horizon (from 58% to

[2]Note that most studies relying on traditional mean/variance optimization assume i.i.d. returns.

[3]Heaney and Sriananthakumar (2012) highlight in particular how the correlation between real estate returns and capital markets is time-varying.

[4]Indeed, real estate markets present many specificities such as transaction costs, lack of liquidity (as illustrated - among other things - by the marketing period risk and the rental vacancy rate). It is also well documented that (the observed or reported) direct (private) real estate returns exhibit autocorrelation (see, among others, Geltner (1991) who deal with smoothing issue and Barkham and Geltner (1995) who point out that returns are not i.i.d.).

[5]For example, as noted by Rehring (2012): "high transaction costs imply expected real estate returns, per period, that are much higher in the long run than in the short run." Regarding the volatility, Rehring (2012) examines the U.K. real estate market and shows that the conditional standard deviation of commercial real estate returns depends on the investment maturity as for usual stocks. In other words, Rehring (2012) internalizes the transaction costs both in the return and volatility modelling.

87% for 20 years horizon). Pagliari (2017) examines real estate's role in institutional mixed-asset portfolios using both private and public real estate indices. He emphasizes the role of the auto-correlation of private-market asset returns. Pagliari (2017) shows that, compared to public-market asset classes that look like random walks, the annualized long-horizon volatility decreases less slowly and long-horizon correlation with most public-market assets increases. Finally, he uses a decreasing long-horizon volatility and suggests an allocation to real estate of somehow 10% to 15% for long term investors.[6] Therefore, as emphasized by Campbell and Viceira (2005) in the general case[7] and by MacKinnon and Al Zaman (2009) in the real estate context, it is necessary to estimate and to take account of the mean-reverting term-structure of expected returns and volatility risks for stocks, bonds, cash and real estate assets.

In this paper, following the aforementioned authors, we analyze the mixed-asset portfolio problem based on four basic financial assets: a money market account (the cash), a bond with constant maturity, a real estate asset and a financial stock index. We assume that the investor seeks to maximize the expected utility of her terminal wealth. According to previous comments on the markets properties and investors behaviors towards risk, we take account of various markets features as follows: First, we do no longer consider static investment strategies (i.e. "buy and hold") but instead dynamic strategies. Indeed, dynamic portfolio optimization ("continuous-time rebalancing") allows to better take account of the financial structure, in particular of the information flows; second, such modelling enables to consider different forms of term structures, for example, a mean-reverting term structure for both excess expected return and volatility; third, we consider a bond with constant maturity.[8]

In this framework, we contribute by providing a general solution to the portfolio allocation problem taking account on one hand of the utility of investors (and thus their risk aversion) and of their time horizons and, on the other hand, by taking account of mean-reverting properties for both expected returns and volatilities term structures. As in Wachter (2002), we assume the market completeness, but to take account simultaneously of the mean-reverting properties of expected returns, volatilities and correlations, we introduce deterministic time-varying functions to model both excess returns and volatilities. As shown in the empirical and numerical sections, such functions correspond usually to negative exponentials with respect to time.[9]

[6]All the previous authors find decreasing volatilities function for real estate and thus real estate is less risky for longer-horizon investors. Similar results are provided by Baroni *et al.* (2008) for the French residential market in Paris.

[7]Campbell and Viceira (2005) examine the U.S. market and show that, contrary to cash, stock returns are mean reverting. This means that the long-term volatility of stock returns (per period) is lower than the short-term return volatility. They find also that bond returns are slighty mean reverting.

[8]As discussed by Bajeux-Besnainou *et al.* (2001), the introduction of constant duration bonds allows to get a bond/stock ratio which increases with time when assuming that returns follow geometric Brownian motions.

[9]Wachter (2002) considers only mean-reverting property of the drift of one single as-

The intertemporal optimization problem is solved by using the martingale approach. To model the multifactor term structure avoiding any arbitrage opportunity, we extend the model introduced by Chiarella *et al.* (2016). Using the martingale approach developed by Cox and Huang (1989), we provide explicit solutions for the optimal portfolio values and the associated portfolio weights. We detail in particular the logarithm, CARA and CRRA cases. Such approach leads to more realistic results. To illustrate these features, we conduct a numerical analysis from Rehring (2012) estimates.

The proposed model contributes to the literature in portfolio allocation and shows how the term structure to maturities of real estate must be taken into account. This specific term structure comes first from the tangible aspect of real estate and second from liquidity constraints inherent to this asset class (all this also has an impact on transaction costs: between 5% and 10% on real estate, for example). In particular, these two characteristics explain why the holding period of a portfolio of real estate assets is much higher than that prevailing on equities and thus why this must be taken into account in the modelling process.

The contribution of the article can be summed-up as follows. First, the current state of research has not been able to reach a consensus about the place of real estate assets in a multi-asset portfolio allocation. Especially, the difference between suggested (10-20%) and actual allocation (7-9%) is still puzzling for academics and practitioners. In this line, our article contributes to the field overall knowledge as our proposed allocation model explains the empirical observed weights. Note that our contribution is particularly appealing considering the investment horizon that comes into play: the longer the investment horizon, the larger the proportion of real estate assets (due to the term structure of returns, volatilities and correlations). Second, the paper proposes a dynamic portfolio optimization model that accounts for the whole real estate term structure. In particular, we derive explicit solutions that consider investment horizon and holding period. The last and not the least this research contributes to the real estate literature in the sense that it justifies why in a context of low interest rate environment, the real estate allocation may increase as it allows capturing de-correlation and liquidity premiums.

The paper is organized as follows. First, we introduce the financial market modelling. In particular, we detail the multi factor model that describes the

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set. This paper introduces a specific Ornstein-Uhlenbeck process to model the instantaneous Sharpe ratio (but with a constant volatility) which is driven by the same Brownian motion as for the dynamics of the risky asset. This allows the financial market completeness. In a discrete-time setting, Campbell and Viceira (2005) introduce a vector autoregressive (VAR) model, which justifies the mean-reverting property of the drift. However, such approach does not lead to exact explicit solutions and moreover volatility is also assumed to be constant. They deal also with only one single asset. When dealing with multi asset allocation, time-varying correlation must be also taken into account. Additionally, looking at financial data, return volatilities are also mean reverting. Finally, in the numerical section, note that all excess expected returns per year and standard deviations per year look like negative exponential functions of time to maturity. Thus, they correspond to the expectations of stochastic mean-reverting processes such as the Ornstein-Uhlenbeck process.

bond and the real estate asset dynamics precluding any arbitrage opportunity. We then explicitly compute the risk-neutral density and determine all the asset risk premia. Then, we provide the solution of the optimization problem, for quite general utility functions and especially for the logarithm, CARA and CRRA cases. Finally, we illustrate the mean-reversion properties of assets and compute numerically the solution for the CRRA case. As a by-product, we show how to calibrate our continuous-time model to cumulative data over time as those of Rehring (2012). We thus analyze the general behavior of the various portfolios. Some of the technical proofs are relegated to appendices.

2 The Financial Market

To take account of various markets features as mentioned previously, we first generalize the framework considered in Chiarella *et al.* (2016) by introducing a multifactor term structure model with time-varying drifts and volatilities precluding any arbitrage opportunity. It is a generalization of Black and Scholes model and a variant of the Merton (1971) one state variable model. The model we consider assumes normality of log returns. This assumption is not too restrictive, when dealing with long term investment. This allows in particular getting explicit formulas. Since standard bonds with long maturities are not generally available on the financial markets and to be in accordance with popular advice, we assume that the bond has a constant duration, as in Bajeux-Besnainou *et al.* (2001) and Rehring (2012). Note that this general modelling with time-varying drifts and volatilities allows to calibrate the model in order to take account of the mean-reverting properties as documented by Rehring (2012) and Pagliari (2017) (see Appendix C).

2.1 The set of basic assets

The market is assumed to be arbitrage-free and without friction. Financial transactions occur in continuous-time, along a time period $[0, T]$. Four basic assets are available at any time on the market:

- An instantaneously riskless money market fund, the cash, with a price denoted by C;
- A real estate asset P;
- A stock index fund with a price S;
- A bond fund B with constant duration D_B obtained by rolling continuously bonds throughout the investment period $[0,\, T]$. It is denoted by B_D which is a zero-coupon bond with maturity $(t + D)$ at time t.

Since continuous-time rebalancing is allowed, financial markets can be assumed to be complete by introducing three sources of risk. For this purpose, we introduce a multidimensional Brownian motion $W = (W^r,\, W^P,\, W^S)$ to describe the uncertainty of the asset returns. The correlation matrix is given by:

$$\Sigma_c = \begin{bmatrix} 1 & \rho^{r,\,P} & \rho^{r,\,S} \\ \rho^{r,\,P} & 1 & \rho^{P,\,S} \\ \rho^{r,\,S} & \rho^{P,\,S} & 1 \end{bmatrix}. \tag{1}$$

Recall that the predictable compensator $\langle W^a,\, W^b\rangle$ of the product $W^a W^b$ satisfies (see e.g. Jacod and Shiryaev, 2002):

$$d\langle W^a,\, W^b\rangle_t = \rho^{a,\,b} dt.$$

2.2 Basic asset dynamics

- The money market C satisfies:

$$\frac{dC_t}{C_t} = r_t dt. \tag{2}$$

- The bond fund B_D with constant duration D_B is solution of:

$$\frac{dB_{D_B,\,t}}{B_{D_B,\,t}} = (r_t + \theta_t^B(D_B))dt - \beta_r(D_B)a_r dW_t^r, \tag{3}$$

where $\theta^B(D_B)$ is the risk premium of bond B. - The real estate asset P satisfies:

$$\frac{dP_{D_P,\,t}}{P_{D_P,\,t}} = (r_t + \theta_t^P)dt + \sigma_t^P dW_t^P, \tag{4}$$

- The stock index fund with price S is defined by:

$$\frac{dS_t}{S_t} = (r_t + \theta_t^S)dt + \sigma_t^S dW_t^S. \tag{5}$$

Denote by τ the remaining time to maturity T (i.e. $\tau = T - t$). We assume that the real interest rate r_t follows an Ornstein-Uhlenbeck process given by:

$$dr_t = k_r(\overline{r} - r_t)dt + a_r dW_t^r, \tag{6}$$

where the convergence speed k_r and the long term value $\overline{r}$ are positive constants.[10] The bond pricing formula is based on an exponential affine model, as introduced by Duffie and Kan (1996):

$$B(r_t, t, T) = \exp\left[-\alpha(\tau) - \beta_r(\tau) r_t\right], \tag{7}$$

[10] Recall that the Ornstein-Uhlenbeck process is given by:

$$r_t = r_0 e^{-k_r t} + \overline{r}(1 - e^{-k_r t}) + a_r e^{-k_r t}\int_0^t e^{k_r s} dW_s^r.$$

It has been introduced by Vasicek (1977) to model stochastic interest rates (see e.g. Brigo and Mercurio, 2006).

where $\alpha(\tau)$ and $\beta_r(\tau)$ are determined by using the no-arbitrage condition. Due to the normalization, the coefficients satisfy the following terminal conditions at maturity:

$$\alpha(0) = 0 \text{ and } \beta_r(0) = 0.$$

We have:

$$\frac{dB(r_t, t, T)}{B(r_t, t, T)} = \mu^B(t,\, \tau)dt - \beta_r(\tau)a_r dW_t^r,$$

$$\mu^B(t, \tau) = \alpha'(\tau) + \beta_r^{'}(\tau)r_t - \beta_r(\tau)k_r(\overline{r} - r_t) + \frac{1}{2}\left[\beta_r^2(\tau)a_r^2\right].$$

$$\begin{aligned}\theta_t^B &= \alpha'(\tau) + \beta_r^{'}(\tau)r_t - \beta_r(\tau)k_r(\overline{r} - r_t) + \frac{1}{2}\left[\beta_r^2(\tau)a_r^2\right] - r_t \\ \sigma_t^B &= \beta_r(T - t)a_r\end{aligned}$$

In order to obtain the bond price, we use the standard no-arbitrage argument as in Chiarella *et al.* (2016). The financial market is complete. Therefore, there exists a unique risk-neutral probability $\mathbb{Q}$ associated to three market premia, λ_r, λ_P and λ_S which density η with respect to the initial probability $\mathbb{P}$ is given by:

$$\eta_t = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\,|\mathbf{F}_t\right] = \exp\left[-\mathbf{M}_t - \int_0^t A_s ds\right], \tag{8}$$

where

$$\mathbf{M}_t = \textstyle\int_0^t \lambda_{r,s} dW_s^r + \int_0^t \lambda_{P,s} dW_s^P + \int_0^t \lambda_{S,s} dW_s^S, \tag{9}$$

and

$$A_t = \frac{1}{2}\left(\lambda_{r,t}^2 + \lambda_{P,t}^2 + \lambda_{S,t}^2\right) + \lambda_{r,t}\lambda_P \rho^{r,\,P} + \lambda_{r,t}\lambda_{S,t}\rho^{r,\,S} + \lambda_{P,t}\lambda_{S,t}\rho^{P,\,S}. \tag{10}$$

Each of the three basic assets B, P and S must satisfy the following condition: when they are discounted by the nominal money market account C, they must be martingales with respect to the risk-neutral probability $\mathbb{Q}$. This is equivalent to the fact that, when their are multiplied by the Radon-Nikodym density and divided by C, they must be martingales with respect to the historical probability $\mathbb{P}$. It is equivalent to the fact that their bounded variation components are equal to 0. This later condition implies the three following equalities: Condition 1: $\left(\left(B(r_t,\, t,\, T)/\exp\left[\int_0^t r_s ds\right]\right) \times \eta_t\right)_t$ is a $\mathbb{P}$-martingale:

$$\mu^B(t,\, \tau) - r_t + a_r\beta_r(\tau)\left[\lambda_{r,t} + \lambda_{P,t}\rho^{r,\,P} + \lambda_{S,t}\rho^{r,\,S}\right] = 0. \tag{11}$$

Condition 2: $\left(\left(P_t/\exp\left[\int_0^t r_s ds\right]\right) \times \eta_t\right)_t$ is a $\mathbb{P}$-martingale:

$$\mu_t^P - r_t - \sigma_t^P(\lambda_{r,t}\rho^{r,\,P} + \lambda_{P,t} + \lambda_{S,t}\rho^{P,\,S}) = 0. \tag{12}$$

Condition 3: $\left(\left(S_t/\exp\left[\int_0^t r_s ds\right]\right) \times \eta_t\right)_t$ is a $\mathbb{P}$-martingale:

$$\mu_t^S - r_t - \sigma_t^S\left[\lambda_{r,t}\rho^{r,\,S} + \lambda_{P,t}\rho^{P,\,S} + \lambda_{S,t}\right] = 0. \tag{13}$$

From the previous system of equations, we deduce the risk premia of the three basic assets P, B, and S. They are defined as the excess instantaneous expected returns with respect to the interest rate r.

Proposition 1 *The three risk premia θ^P, θ^B and θ^S are given by:*
i) Nominal Bond B: *For fixed duration D_B,*

$$\theta_t^B = \mu^B(t,\, D_B) - r_t = -a_r \beta_r(D_B)\left[\lambda_{r,t} + \lambda_{P,t}\rho^{r,\,P} + \lambda_{S,t}\rho^{r,\,S}\right]. \tag{14}$$

ii) Real estate asset P:

$$\theta_t^P = \mu_t^P - r_t = \sigma_t^P(\lambda_{r,t}\rho^{r,\,P} + \lambda_{P,t} + \lambda_{S,t}\rho^{P,\,S}). \tag{15}$$

iii) Stock price S:

$$\theta_t^S = \mu_t^S - r_t = \sigma_t^S\left[\lambda_{r,t}\rho^{r,\,S} + \lambda_{P,t}\rho^{P,\,S} + \lambda_{S,t}\right], \tag{16}$$

where λ^r, λ^P and λ^S are usually interpreted as the market prices of risk (or factor loadings) associated respectively to the sources of risk W^r, W^P and W^S.

Functions $\alpha(.)$ and $\beta_r(.)$, which are involved in the exponential affine model describe the value of the bond. They are given by:

$$\beta_r(\tau) = \left(\frac{1-\exp(-\tau k_r)}{k_r}\right).$$

and

$$\frac{\alpha(\tau)}{\tau} = a_r\left[\frac{\overline{r}}{a_r} - \frac{\lambda_r + \lambda_P\rho^{r,\,P} + \rho^{r,\,S}\lambda_S}{k_r}\right]\left[1 - \frac{1}{\tau k_r} + \frac{\exp(-\tau k_r)}{\tau k_r}\right]$$
$$-\frac{a_r^2}{2k_r^2}\left[1 - \frac{2\left[1-\exp(-\tau k_r)\right]}{\tau k_r} + \frac{1-\exp(-2\tau k_r)}{2\tau k_r}\right].$$

In this setting, $B(r_t,\, t,\, T)$ can be replicated using the assets C and B_D which span the bond market, as in Bajeux *et al.* (2001) for a simpler case. This dynamic combination of fixed-income securities of different durations is referred to as the *passive immunization* (see Fong, 1990; Fabozzi, 2000). From previous Proposition 1, we can determine the market prices of risk λ_r, λ_P and λ_S. Indeed, introduce the matrix Γ equal to:

$$\begin{bmatrix} -a_r\beta_r(D_B) & -a_r\beta_r(D_B)\rho^{r,\,P} & -a_r\beta_r(D_B)\rho^{r,\,S} \\ \sigma_t^P\rho^{r,\,P} & \sigma_t^P & \sigma_t^P\rho^{P,\,S} \\ \sigma_t^S\rho^{r,\,S} & \sigma_t^S\rho^{P,\,S} & \sigma_t^S \end{bmatrix}. \tag{17}$$

We have:

$$\begin{bmatrix} \theta^B \\ \theta^P \\ \theta^S \end{bmatrix} = \Gamma.\begin{bmatrix} \lambda_r \\ \lambda_P \\ \lambda_S \end{bmatrix}, \text{ which implies } \begin{bmatrix} \lambda_r \\ \lambda_P \\ \lambda_S \end{bmatrix} = \Gamma^{-1}.\begin{bmatrix} \theta^B \\ \theta^P \\ \theta^S \end{bmatrix}.$$

3 Optimal portfolios

Let us introduce the matrix Σ of the rates of return of B, P, S given by:

$$\Sigma_t = \begin{bmatrix} -\beta_r(D_B)a_r & 0 & 0 \\ 0 & \sigma_t^P & 0 \\ 0 & 0 & \sigma_t^S \end{bmatrix}. \tag{18}$$

We recall the standard results about optimal portfolio computation when the financial is complete (see e.g. Prigent, 2007). Portfolio weights are respectively denoted by x_C, x_B, x_P and x_S (with $x_C + x_B + x_P + x_B = 1$). The portfolio value at time t is denoted by V_t. Therefore, the portfolio value V follows the dynamics:

$$\frac{dV_t}{V_t} = [r_t + x_B(t)\theta_t^B + x_P(t)\theta_t^P + x_S(t)\theta_t^S]dt \tag{19}$$

$$+x_B(t)\left[-\beta_r(D_B)a_r dW_t^r\right] + x_P(t)\left[\sigma_t^P dW_t^P\right] + x_S(t)\left[\sigma_t^S dW_t^S\right].$$

The investor's preferences is described by her utility function U, which embeds her risk aversion. Function U is assumed to satisfy usual properties, namely it is strictly increasing, strictly concave and twice continuously differentiable (see e.g. Gollier (2001) for definitions and main properties of utility functions). We consider an investor with an initial capital denoted by V_0. She is assumed to maximize the expected utility over the time horizon T, defined on the portfolio value V_t. Thus, her optimal portfolio weights are the solutions of the following problem:

$$\underset{x_C,\, x_B,\, x_P,\, x_S}{Max} \mathbb{E}\left[U\left(V_T\right)\right].$$

For different utility functions, we determine the optimal portfolio. We show in particular how it depends on the investor's risk aversion. We detail the solutions for the logarithm, HARA and CARA utility functions.

3.1 General result

To solve the optimization problem given V_0, where the set of variables is the collection of all self-financing strategies, Cox and Huang (1989) use the market completeness to propose a change of variables. The new variables are all possible portfolio values at maturity with only one constraint:

$$V_0 = \mathbb{E}_{\mathbb{P}}\left[V_T \frac{\eta_T}{exp\left(\int_0^T r_s\, ds\right)}\right]. \tag{20}$$

Differentiating with respect to the variables $V_T(\omega)$ for each random event ω, leads to the equation:

$$U'\left(V_T\right) = \nu \frac{\eta_T}{exp\left(\int_0^T r_s\, ds\right)},$$

where ν is a Lagrange multiplier determined by the initial investment constraint. Denoting $J(y) = (U')^{-1}(y)$, the optimal solution is given by:

$$V_T^* = J\left(\nu \frac{1}{H_T}\right). \tag{21}$$

where H_T is the numeraire portfolio, equal to:

$$H_T = \left(\frac{\eta_T}{\exp\left(\int_0^T r_s ds\right)}\right)^{-1}. \tag{22}$$

To compute the replicating strategies for a given optimal portfolio, first note that V_t^*/H_t is a martingale. Thus, we obtain the martingality relation:

$$\frac{V_t^*}{H_t} = \mathbb{E}_{\mathbb{P},t}\left[V_T^*/H_T\right]. \tag{23}$$

Therefore, it is necessary to compute conditional expectations of quantities which are functions of V_t^*/H_t. For this purpose, the conditional expectations of the numeraire portfolio have to be used. We first consider the logarithm utility case. For the other cases, note that the optimal portfolios are functions of the optimal logarithm portfolio.

3.1.1 The logarithm case

The logarithm utility function is defined by $U(x) = \ln(x)$ for $x > 0$, so that the absolute risk aversion is $-U''(x)/U'(x) = 1/x$ and the relative risk aversion is constant and equal to $-xU''(x)\ /U'(x) . = 1$. In that case, the optimal portfolio is called the numeraire portfolio (see Long (1990) or the growth-optimal portfolio (see Merton, 1992). Its value at maturity T, denoted by $V_T^{\ln}$, is given by:

$$V_T^{\ln} = V_0\ H_T. \tag{24}$$

Note that $\left(\int_t^T r_s\ ds\right)$ has a Gaussian distribution. Thus, the ratio

$$[(H_T/H_t)]^z$$

has a Lognormal distribution for any power $z \neq 0$. This ratio is equal to $\exp[N_z(t, T)]$ where $N_z(t, T)$ has a Gaussian distribution. Therefore, the conditional expectation $E_t\left[[(H_T/H_t)]^z\right]$ is defined by:

$$\mathbb{E}_t[[(H_T/H_t)]^z] = \exp[E(N_z)(t, T) + \frac{1}{2}Var(N_z)(t, T)], \tag{25}$$

where: (see Appendix A)

$$\mathbb{E}(N_z)(t, T) = z\Phi_{(t,T)} \text{ and } Var(N_z)(t, T) = z^2\Psi_{(t,T)},$$

$$\Phi_{(t,T)} = (T-t) \times \left(\overline{r} + (r_t - \overline{r}) \frac{\beta_r(T-t)}{T-t} + \frac{1}{T-t} \int_t^T A(s)ds \right),$$

and

$$\Psi_{(t,T)} =$$
$$\frac{a_r^2}{k_r^2} \left[(T-t) + \tfrac{1}{2k_r} \left[1 - \exp(-2(T-t)k_r)\right] - 2\beta_r(T-t) \right]$$
$$+2\int_t^T A(s)ds + 2a_r \int_t^T \left(\lambda_{r,s} + \lambda_{P,s}\rho^{r,P} + \lambda_{S,s}\rho^{r,S} \right) \left[\tfrac{1-e^{-(T-s)k_r}}{k_r} \right] ds,$$

with

$$A(t) = \frac{1}{2} \left(\lambda_{r,t}^2 + \lambda_{P,t}^2 + \lambda_{S,t}^2 \right) + \lambda_{r,t}\lambda_{P,t}\rho^{r,\,P} + \lambda_{r,t}\lambda_{S,t}\rho^{r,\,S} + \lambda_{P,t}\lambda_{S,t}\rho^{P,\,S}.$$

Denote respectively by $\widehat{\alpha}_{N,(t,\,T)}$ and $\widehat{\beta}_{N,(t,\,T)}$ the conditional expectations:

$$\mathbb{E}_t[(H_t/H_T)] \text{ and } \mathbb{E}_t[H_t \ln((H_t/H_T))/H_T].$$

Then we deduce:

$$\widehat{\alpha}_{N(t,\,T)} = \exp[-\Phi_{(t,\,T)} + \frac{1}{2}\Psi_{(t,\,T)}] \text{ and } \widehat{\beta}_{N(t,\,T)} = \widehat{\alpha}_{N(t,\,T)} \times \left(\Phi_{(t,\,T)} - \Psi_{(t,\,T)} \right).$$

In order to simplify notations, we respectively denote by $E(N_z)(T)$, $Var(N_z)(T)$, Φ_T, Ψ_T, $\widehat{\alpha}_{N,\,T}$ and $\widehat{\beta}_{N,\,T}$ the values of $E_t(N_z)(0,\,T)$, $Var(N_z)(0,\,T)$, $\Phi_{(0,\,T)}$, $\Psi_{(0,\,T)}$, $\widehat{\alpha}_{(0,\,T)}$ and $\widehat{\beta}_{(0,\,T)}$. For the logarithmic case, the optimal weights are determined as follows. The process $\left(\frac{V^{\ln}}{H}\right)_t$ is a $\mathbb{P}$-martingale. Therefore, we have:

$$\frac{V_t^{\ln}}{H_t} = \mathbb{E}_{\mathbb{P},t}\left[\frac{V_T}{H_T}\right],$$

which implies:

$$V_t^{\ln} = V_0 H_t.$$

But, the portfolio value V is solution of the following (SDE):

$$\frac{dV_t^{\ln}}{V_t} = [r_t + x_B(t)\theta^B + x_P(t)\theta^P + x_S(t)\theta^S]dt$$

$$+x_B(t)\left[-\beta_r(D_B)a_r dW_t^r\right] + x_P(t)\left[\sigma_t^P dW_t^P\right] + x_S(t)\left[\sigma_t^S dW_t^S\right].$$

Additionally, we have:

$$dV_t^{\ln} = V_0 dH_t \text{ and } dH_t = H_t\left[r_t dt + \mathbf{M}_t + 2A(t)dt\right].$$

To determine the weights, we must identify the martingale parts (the three stochastic Brownian integrals). We obtain the following system:

$$\left\{ \begin{array}{lcc} \lambda_{r,t} & = & -x_B(t)\beta_r(D_B)a_r \\ \lambda_{P,t} & = & x_P(t)\sigma_t^P \\ \lambda_{S,t} & = & x_S(t)\sigma_t^S \end{array} \right.$$

This relation is equivalent to:[11] (denote by Λ the factors vector and X the weighting vectors on B, P and S)

$$\Lambda = {}^t\Sigma X, \text{ which is also equivalent to } X = \left({}^t\Sigma\right)^{-1}\Lambda.$$

Therefore, we deduce the optimal weights at any time t.

Proposition 2 *For the logarithmic case, the optimal weights are given by:*

$$\left\{\begin{array}{lcc} x_B(t) = h_{B,t} & = & -\frac{\lambda r,t}{\beta_r(D_B)a_r} \\ x_P(t) = h_{P,t} & = & \frac{\lambda_{P,t}}{\sigma_t^P} \\ x_S(t) = h_{S,t} & = & \frac{\lambda_{S,t}}{\sigma_t^S} \\ x_C(t) = h_{C,t} & = & 1 - x_B(t) - x_P(t) - x_S(t) \end{array}\right. \tag{26}$$

In the growth-optimal portfolio, h_C, h_B, h_P and h_S represent respectively the weights invested on the nominal money account, on the value of the bond with constant maturity, on the real asset with constant duration and on the stock.

3.1.2 CARA utility function

Assume now that the investor has a constant absolute risk aversion case which corresponds to the exponential utility function $\widehat{U}_a(x) = -\exp(-ax)/a$ where a is the absolute risk aversion ($a > 0$). The optimal portfolio value for the exponential utility function, $V_T^{\exp}$, is a function of the numeraire portfolio given by:

$$V_T^{\exp} = J\left(\upsilon\frac{1}{H_T}\right),$$

which is equivalent to:

$$V_T^{\exp} = -\frac{1}{a}\left[\ln(\lambda) + \ln(1/H_T)\right] = -\frac{1}{a}\left[\log(\upsilon) + \ln(\eta_T \exp(-\textstyle\int_0^T R_s\, ds))\right],$$

where υ is solution of:

$$\ln(\upsilon) = \frac{-aV_0 - \mathbb{E}\left[(1/H_T)\ln(1/H_T)\right]}{\mathbb{E}\left[(1/H_T)\right]}.$$

The previous condition implies that the optimal value $V_T^{\exp}$is equal to:

$$V_t^{\exp} = H_t\mathbb{E}_t\left[(1/H_T)\left(\frac{1}{a}A(V_0) + \ln(H_T)\right)\right],$$

[11] Note also that we can determine the optimal weighting vector X from the vector of risk premia θ itself since we have:

$$X = \left({}^t\Sigma\right)^{-1}\Lambda = \left({}^t\Sigma\right)^{-1}\Gamma^{-1}\theta.$$

where:

$$A(V_0)=\frac{aV_0+\mathbb{E}\left[(1/H_T)\ln(1/H_T)\right]}{\mathbb{E}\left[(1/H_T)\right]}=\frac{aV_0+\widehat{\beta}_{N,T}}{\widehat{\alpha}_{N,T}}.$$

Using the martingality relation, the value $V_T^{\exp}$ is given by:

$$V_t^{\exp}=H_t\mathbb{E}_t\left[(1/H_T)\left(\frac{1}{a}A(V_0)+\ln(H_T)\right)\right].$$

Finally, we obtain:

Proposition 3 *For the CARA case, the optimal portfolio value at any time t of the management period is given by:*

$$V_t^{\exp}=\left[A(V_0)\widehat{\alpha}_{N,(t,T)}+\frac{1}{a}\widehat{\beta}_{N,(t,T)}\right]+\frac{1}{a}\ln(H_t)\widehat{\alpha}_{N,(t,T)}. \tag{27}$$

Using the same approach as for the numeraire portfolio, we can determine the optimal weights for the CARA case, as shown in Appendix B.

3.1.3 HARA utility function

Consider utility function which has a hyperbolic absolute risk aversion (HARA). This family of utility functions can be written as follows:

$$U_\gamma(x)=\left(\frac{\gamma}{1-\gamma}\right)\left(\frac{x-x^*}{\gamma}\right)^{1-\gamma}. \tag{28}$$

where γ and x^* are two parameters that cannot be both negative. The risk aversion is an increasing function of the absolute value of γ. It includes the CRRA utility function ($x^*=0$), in particular the logarithmic case for $\gamma=1$. The optimal portfolio at maturity T,$V_{(\gamma,T)}^{HARA}$, is given by:

$$V_{(\gamma,T)}^{HARA}=x^*+\left(\frac{V_0-x^*\widehat{\alpha}_{N,T}}{E[(H_T)^{\left(\frac{1-\gamma}{\gamma}\right)}]}\right)(H_T)^{\left(\frac{1}{\gamma}\right)}. \tag{29}$$

This expression can be interpreted as follows: the optimal portfolio is a combination of a CRRA fund with γ parameter and a zero-coupon bond with nominal x^* at time T. We consider the special case of CRRA utility function. This means that the investor does not impose the guarantee constraint $V_T\geq x^*$ at maturity. Thus, this case corresponds to $x^*=0$. The optimal portfolio weights for the CRRA utility function are determined from the following relations (see Appendix B for details): (denote $z=\frac{1-\gamma}{\gamma}$)

$$\begin{cases}\frac{\lambda_{r,t}}{\gamma}+za_r\beta_r(T-t) & =\delta_B(t)= & -x_B(t)\beta_r(D_B)a_r\\ \frac{\lambda_{P,t}}{\gamma} & =\delta_P(t)= & x_P(t)\sigma_t^P\\ \frac{\lambda_{S,t}}{\gamma} & =\delta_S(t)= & x_S(t)\sigma_t^S\end{cases}$$

Therefore, we deduce the optimal weights at any time t.

Proposition 4 *The optimal weights for the CRRA case are given by:*

$$\begin{cases} x_B^{CRRA}(t) &= & -\frac{\delta_B(t)}{\beta_r(D_B)a_r} \\ x_P^{CRRA}(t) &= & \frac{\delta_P}{\sigma_t^P} \\ x_S^{CRRA}(t) &= & \frac{\delta_S}{\sigma_t^S} \\ x_C^{CRRA}(t) &= & 1 - x_B^{CRRA} - x_P^{CRRA} - x_S^{CRRA} \end{cases} \tag{30}$$

Remark 5 *(Two-funds separation) The optimal CRRA weights X^{CRRA} can be decomposed as follows:*

$$X^{CRRA} = \frac{1}{\gamma}X^{Log} + (1 - \frac{1}{\gamma})X^{Conservative}, \tag{31}$$

where X^{Log} is the optimal portfolio for the logarithm case, and $X^{Conservative}$ corresponds to the optimal portfolio for an infinite relative risk aversion γ. Here, this latter portfolio is equal to:

$$X^{Conservative} = \begin{pmatrix} \beta_r(T-t)/\beta_r(D_B) \\ 0 \\ 0 \end{pmatrix}. \tag{32}$$

Remark 6 *The ratios of asset weights are immediately deduced from Proposition 4. Indeed, for example both bond/stock and real estate/stock ratios are respectively given by:*

$$\begin{aligned} x_B(t)/x_S(t) &= h_{B,t}/h_{P,t} + (\gamma - 1)\beta_r(T-t)/(\beta_r(D_B)/h_{P,t}), \\ x_P(t)/x_S(t) &= h_{P,t}/h_{S,t}. \end{aligned} \tag{33}$$

In Bajeux-Besnainou et al. (2001), when there is no real estate asset and when risky assets follow geometric Brownian motions, the first above equation shows that the bond/stock ratio is increasing with respect to time.

To study the corresponding portfolio return distribution, we briefly recall how to compute its expectation and its cumulative distribution function. The portfolio return is given by:

$$V_T^{CRRA} = \frac{V_0}{\mathbb{E}\left[H_T^{\frac{1-\gamma}{\gamma}}\right]} (H_T)^{\frac{1}{\gamma}} .$$

The mean of the return per year is given by:

$$\mathbb{E}\left[\left(\frac{V_T^{CRRA}}{V_0}\right)^{\frac{1}{T}}\right] = \frac{\mathbb{E}\left[\left[(H_T)^{\frac{1}{\gamma}}\right]^{\frac{1}{T}}\right]}{\left(\mathbb{E}\left[(H_T)^{\frac{1-\gamma}{\gamma}}\right]\right)^{\frac{1}{T}}}.$$

But we have:

$$\mathbb{E}\left[H_T^{\frac{1-\gamma}{\gamma}}\right] = \exp\left[z\Phi_{(0,T)} + \frac{1}{2}z^2\Psi_{(0,T)}\right], \text{ with } z = \frac{1-\gamma}{\gamma}.$$

Consider now the real portfolio return per year $(\frac{V_T^{(r)CRRA}}{V_0})^{\frac{1}{T}}$. We get:

$$\mathbb{P}\left[\left(\frac{V_T^{CRRA}}{V_0}\right)^{\frac{1}{T}} \leq x\right] =$$

$$\mathbb{P}\left[\exp\left[\frac{1}{\gamma}N_1(0,T)\right] \leq x^T \exp\left[z\Phi_{(0,T)} + \frac{1}{2}z^2\Psi_{(0,T)}\right]\right] =$$

$$\mathbb{P}\left[N_1(0,T) \leq \gamma\left(\log(x^T) + z\Phi_{(0,T)} + \frac{1}{2}z^2\Psi_{(0,T)}\right)\right] =$$

$$N\left[\frac{\gamma T\log(x) - \gamma\Phi_{(0,T)} + \frac{1}{2}\frac{(1-\gamma)^2}{\gamma}\Psi_{(0,T)}}{\sqrt{\Psi_{(0,T)}}}\right], with\ z = \frac{1-\gamma}{\gamma}.$$

The expected real return per year T is given by:

$$\mathbb{E}_0\left[\left(\frac{V_T^{CRRA}}{V_0}\right)^{\frac{1}{T}}\right] = \mathbb{E}_0\left[\frac{(H_T)^{\frac{1}{\gamma T}}}{\mathbb{E}\left[(H_T)^{\frac{1-\gamma}{\gamma T}}\right]}\right] = \frac{\exp\left[\frac{1}{\gamma T}\Phi_{(0,T)} + \frac{1}{2}\left(\frac{1}{\gamma T}\right)^2\Psi_{(0,T)}\right]}{\exp\left[\frac{1-\gamma}{\gamma T}\Phi_{(0,T)} + \frac{1}{2}\left(\frac{1-\gamma}{\gamma T}\right)^2\Psi_{(0,T)}\right]},$$

from which we deduce:

$$\mathbb{E}_0\left[\left(\frac{V_T^{CRRA}}{V_0}\right)^{\frac{1}{T}}\right] = \exp\left[\frac{1}{T}\Phi_{(0,T)} - \frac{1}{T^2}\left(\frac{1}{2} - \frac{1}{\gamma}\right)\Psi_{(0,T)}\right].$$

4 Numerical illustration

We illustrate numerically the theoretical solution from Rehring (2012), who consider annual dataset from 1965 to 2008 for the U.K. market. The cash return corresponds to U.K. three-month treasury bill, the bond yield to yield of Barclays gilt index, the stock to Barclays equity index and the real estate is constructed as described in Rehring (2012) from unsmoothed log real capital returns.

4.1 Excess return per year and instantaneous excess return (term structure)

Recall that the short-term interest rate is assumed to follow an Ornstein-Uhlenbeck process as introduced by Vasicek (1977) (see Relation 6). Using Rehring (2012) data on UK three-month treasury bill, the estimates of the short-term parameters are given by:

$$a_r = 0.03, k_r = 0.2; r_0 = 1.8\%; \overline{r} = 1.8\%.$$

In a first step we calibrate the excess cumulated expected returns per year. We find that approximations by negative exponentials of type $\alpha_\mu - \beta_\mu exp(-\lambda_\mu t)$ provide a good fit to observed data (see Figures 1 and 3). For the real estate P and the stock S, we get respectively:

$$\begin{aligned}
\alpha_\mu^P &= 0.04; \beta_\mu^P = 0.08; \lambda_\mu^P = 0.4, \\
\alpha_\mu^S &= 0.061; \beta_\mu^S = 0.007; \lambda_\mu^S = 0.061.
\end{aligned}$$

Then, to fit instantaneous excess return term structure, we begin by substracting the expectation of the short-term interest rate, namely:

$$\mathbb{E}[r_t] = \overline{r} - (\overline{r} - r_0) exp(-k_r t).$$

Finally, we apply results of Appendix C to identify the instantaneous excess return of both the real estate asset and the stock, as shown in Figures 1, 2, 3 and 4. Figures 1 and 3 present the calibration of expected return per year of the real estate and stocks asset respectively and Figures 2 and 4 present the calibration of the drift term structure of the real estate and stocks asset respectively. Recall here that the calibrations are based on the empirical results of Rehring (2012). These figures raise number of comments. First, our model allows to accurately fit the empirical analysis of Rehring (2012), which highly contributes to attest the quality of our analysis. Note that the model is more closely related to the expected return per year than to the drift term structure. This can be attributed to both the complexity of the term structure on the long run and the limited amount of data. Second, the difference between real estate and stocks assets is obvious. Indeed, when the expected return per year is decreasing over horizon for stocks, it is increasing for real estate assets. Third and finally, it is interesting to concentrate on returns and expected return numbers. In particular, real estate returns (and expected returns) are negative on the short-term period but strongly increase after 3-4 years (see Figures 1 and 2) indicating how this asset is sensitive to the holding period and therefore highlighting the usefulness of our modelling. This, in addition, is the contrary for stocks as shown in Figures 3 and 4.

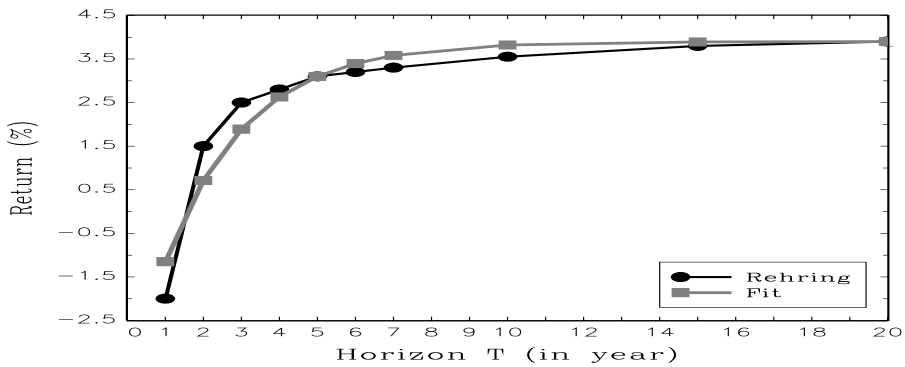

Figure 1: Calibration of the expected return per year of the real estate asset

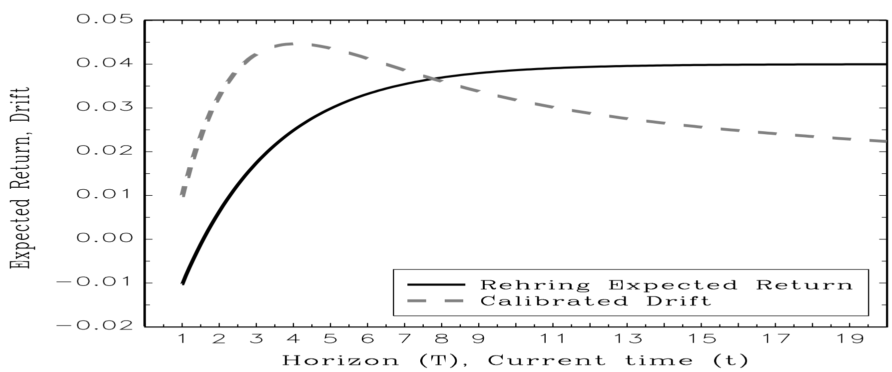

Figure 2: Calibration of the drift term structure of the real estate asset

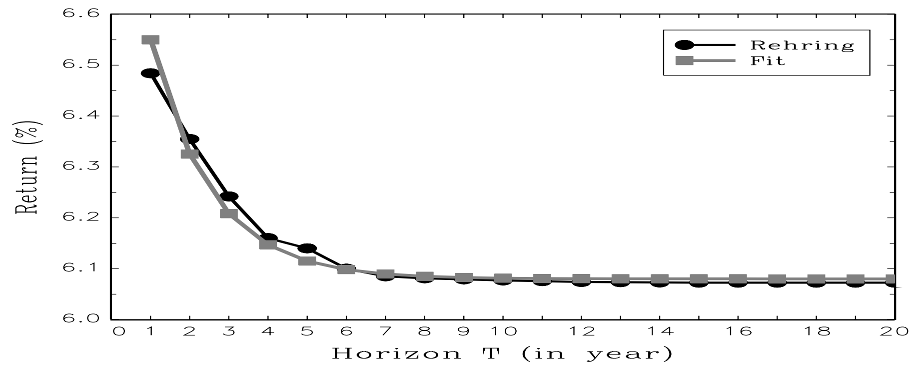

Figure 3: Calibration of the expected return per year of the stock

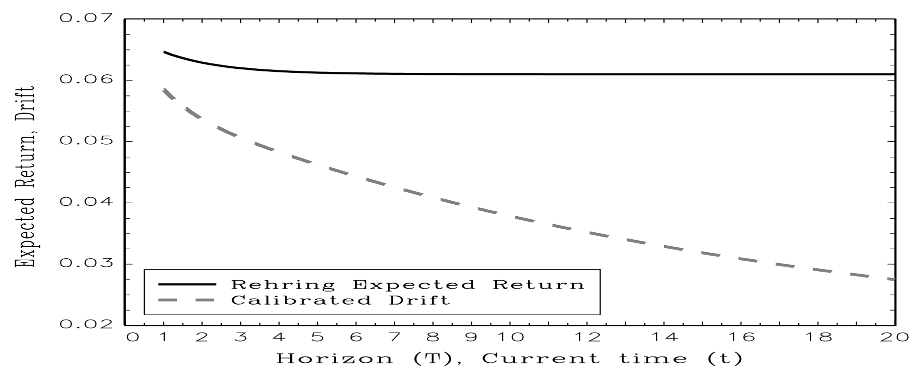

Figure 4: Calibration of the drift term structure of the stock

4.2 Standard deviation per year and volatility (term structure)

To fit the standard deviation per year, we can also use negative exponentials of type $\alpha_\sigma - \beta_\sigma exp(-\lambda_\mu t)$. We get:

$$\begin{aligned} \alpha_\sigma^P &= 0.085; \beta_\sigma^P = -0.12; \lambda_\sigma^P = 0.45, \\ \alpha_\sigma^S &= 0.134; \beta_\sigma^S = -0.13; \lambda_\sigma^S = 0.455. \end{aligned}$$

Applying results of Appendix C, we determine the volatility of both the real estate asset and the stock, as shown in Figures 5 and 6.

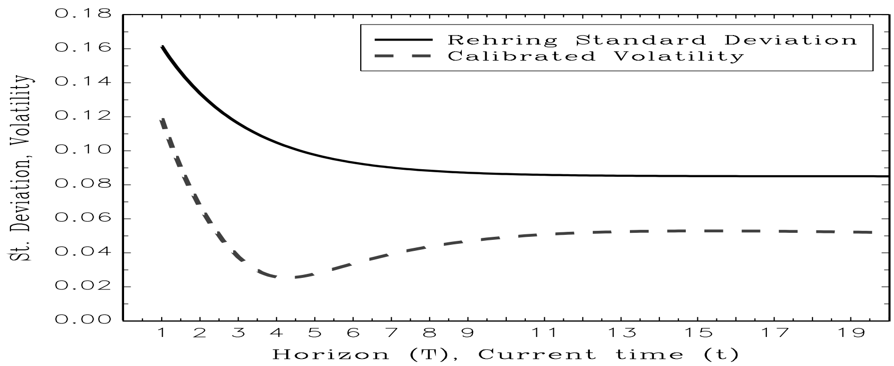

Figure 5: Calibration of the standard deviation per year T and the volatility term structure of the real estate asset, function of current time t

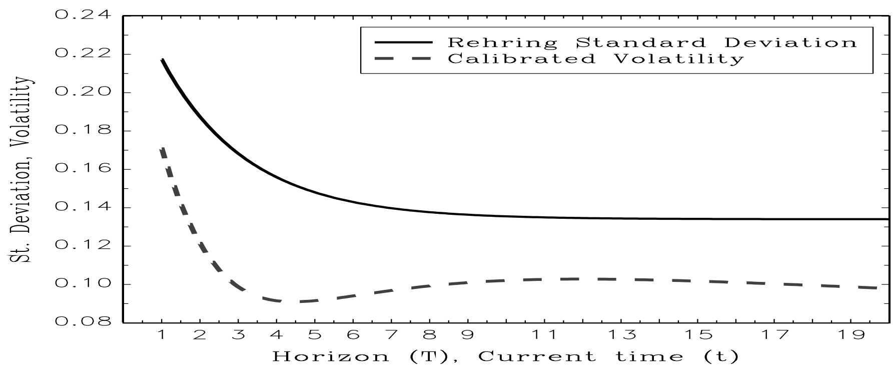

Figure 6: Calibration of the standard deviation per year T and the volatility term structure of the stock, function of current time t

4.3 Correlation term structure

We search to calibrate the correlations of the three Brownian motions, namely the three parameters $\rho^{r,P}$, $\rho^{r,S}$ and $\rho^{P,S}$. Note that these correlations do not depend on current time t during a given management period $[0,T]$. But, for fixed horizon T, we can calibrate the constant parameters ${\rho_T}^{r,P}$, ${\rho_T}^{r,S}$ and ${\rho_T}^{P,S}$ to actual data such as those of Rehring (2012), in particular from annualized correlations of cumulated asset returns (see Appendix C). Results are provided in Figures 7, 8 and 9.

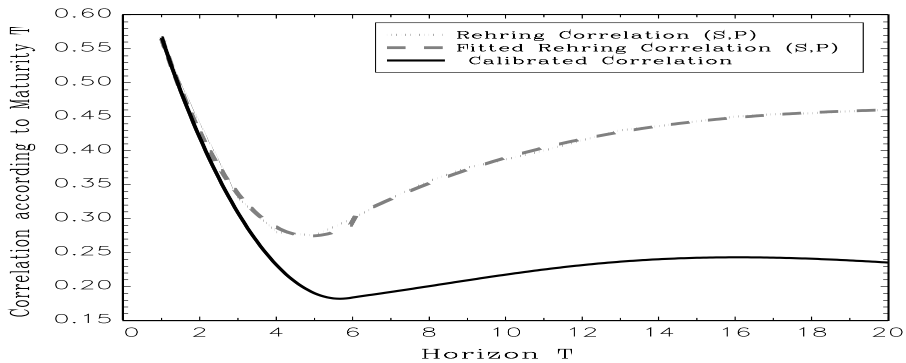

Figure 7: Calibration of the correlation of real estate and stock

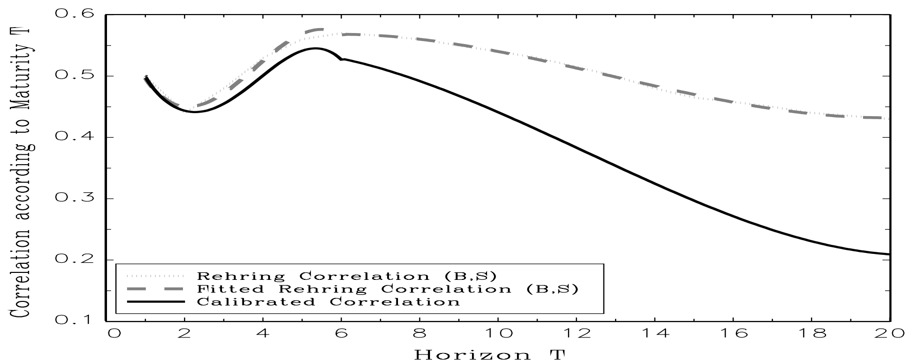

Figure 8: Calibration of the correlation of bond and stock

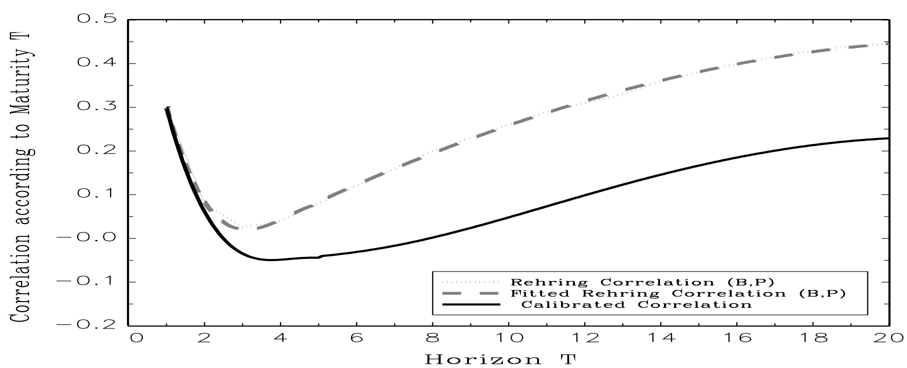

Figure 9: Calibration of correlation of bond and real estate

4.4 Optimal weights (CRRA case)

For the numerical illustrations of how the optimal portfolio allocation depends on both the current time t and the time horizon T, we consider three types of investors with CRRA utilities: the first one is aggressive ($\gamma = 3$), the second one is more moderate ($\gamma = 5$), the third one is conservative ($\gamma = 15$). We apply results of Proposition 4 to illustrate the optimal weights for the CRRA case. We begin by providing the optimal weights horizon, namely $T = 20$ years. Figures 10, 11 and 12 show that the optimal weight on the real estate asset is always increasing with respect to current time, as in Rehring (2012) for $t > 3$ years. At maturity ($t = T = 20$), it reaches respectively about 85% for $\gamma = 3$, 50% for $\gamma = 5$ and 17% for $\gamma = 15$. Both the weights on the bond and the stock are decreasing.

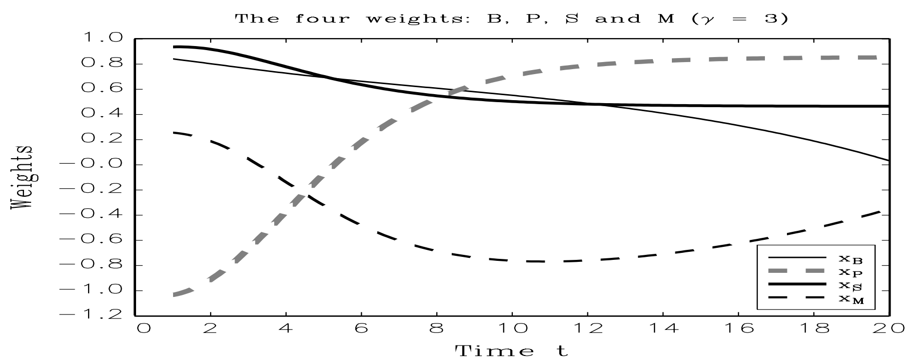

Figure 10: Weights for maturity of 20 years and relative risk aversion = 3

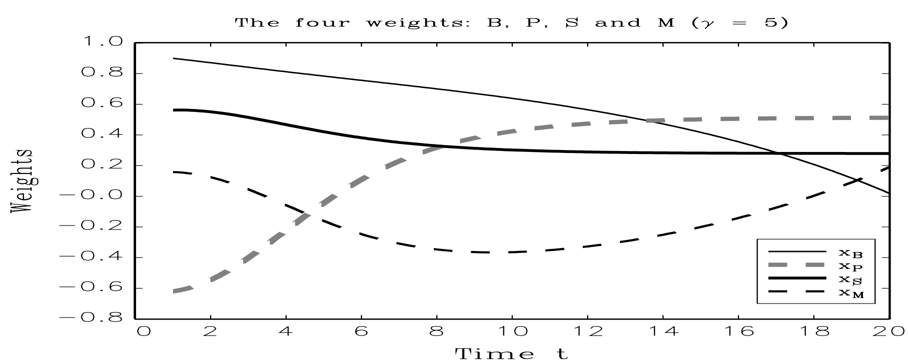

Figure 11: Weights for maturity of 20 years and relative risk aversion = 5

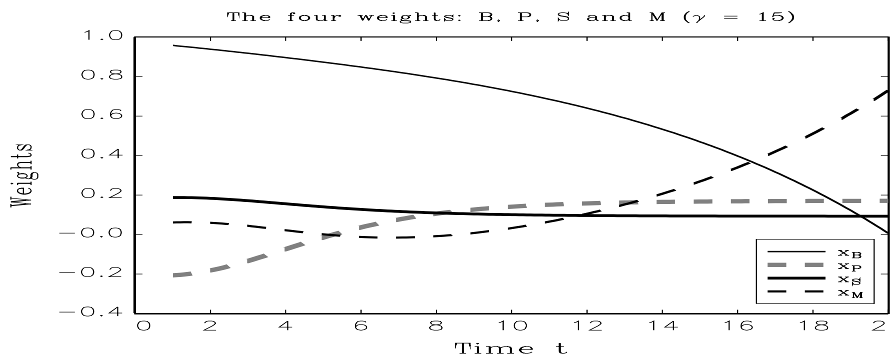

Figure 12: Weights for maturity of 20 years and relative risk aversion = 15

Allocating assets means allocating risks/returns trade-off. One of the interests of real estate assets is that they behave differently from bonds or equities, and are generally said to be poorly correlated with other standard assets, which means that they can be combined advantageously (see Eichholtz, 1996).

Indeed, real estate assets exhibit a low liquidity since real estate obviously cannot be sold quickly without high losses. Indeed this liquidity induces some modelling specificities.

As it can be seen from Figures (1, 2) and (3, 4), both the expected return per year and the drift term structure are increasing with respect to time for the real estate asset while it is the converse for the stock.

Additionally, looking at Figures (5, 6), both their standard deviations per year and their volatility term structures have a similar behavior with respect to time. This explains why the allocation on real estate asset is increasing with respect to time while it is not the case for the stock.

We examine now the optimal weights for a shorter horizon, namely $T = 10$ years. Figures 13, 14 and 15 show that the optimal weight on the real estate asset is still increasing with respect to current time.

At maturity ($t = T = 10$), it reaches respectively about 65% for $\gamma = 3$, 40% for $\gamma = 5$ and 12% for $\gamma = 15$. Both the weights on the bond and the stock are also still decreasing. Compared to the maturity of 20 years case, we note that, at a given current time t, weights at $t = 10$ years are not exactly equal to those for an horizon of 10 years. This is due to the term structure. For example, when dealing with a maturity of 20 years, the weights computed at $t = 10$ years are respectively equal to 70% for $\gamma = 3$, 43% for $\gamma = 5$ and 16% for $\gamma = 15$. Thus, they are slightly higher than previous ones.

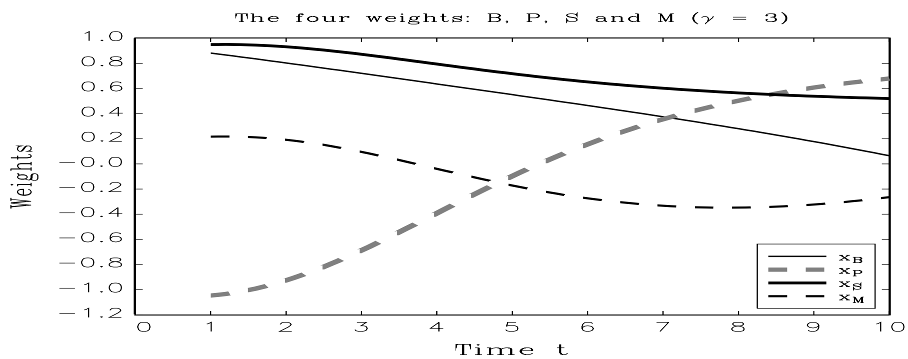

Figure 13: Weights for maturity of 10 years and relative risk aversion = 3

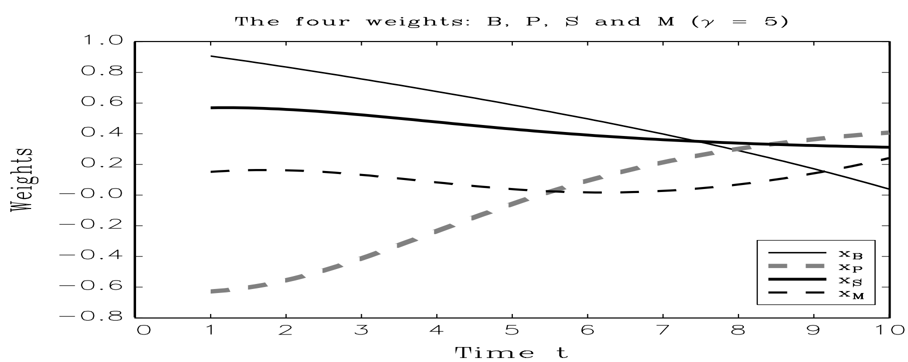

Figure 14: Weights for maturity of 10 years and relative risk aversion = 5

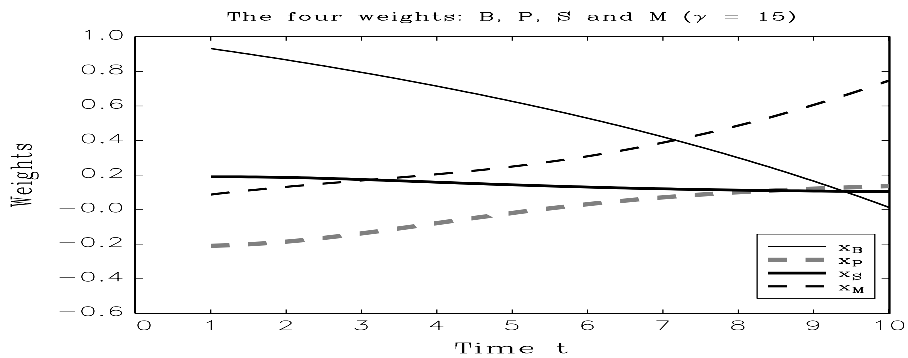

Figure 15: Weights for maturity of 10 years and relative risk aversion = 15

For the numerical illustrations of how the optimal portfolio allocation depends on the relative risk aversion γ, we consider a time horizon equal to 20 years and four current times, namely $t = 2$, 5, 10 and 20 years.

Figures 16 and 17 show in particular that the allocation on the real estate asset can be increasing with respect to relative risk aversion, provided that the current time is relatively small compared to the time horizon.

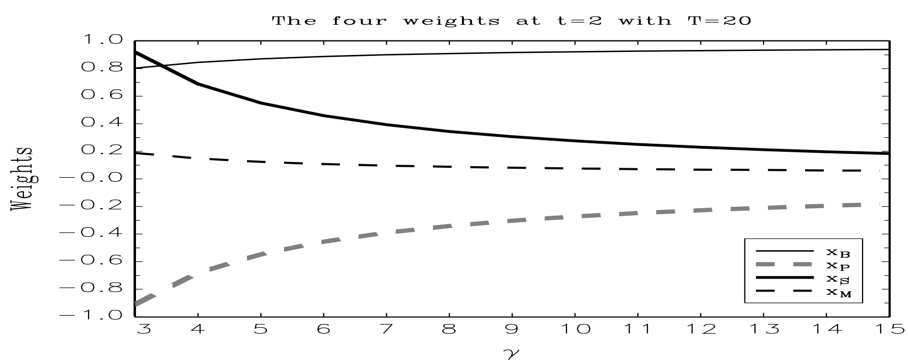

Figure 16: Weights for maturity of 20 years evaluated at time t=2 years

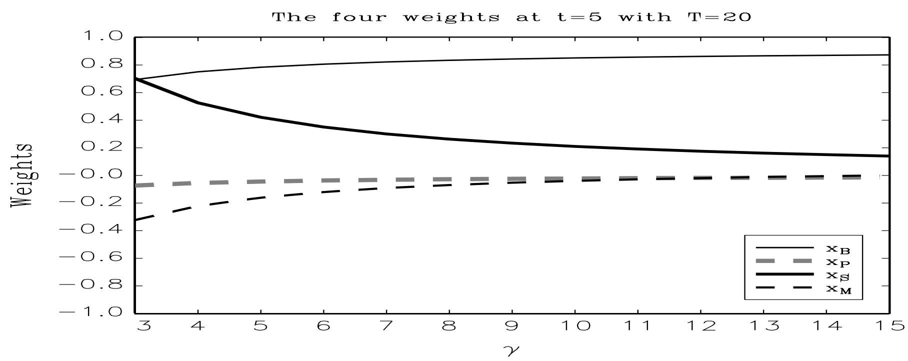

Figure 17: Weights for maturity of 20 years evaluated at time t=5 years

Figures 18 and 19 show that the allocation on the real estate asset is decreasing with respect to relative risk aversion, provided that the current time is relatively high compared to the time horizon. This latter property is more in accordance with the findings of Rehring (2012) who considers static allocations for several time horizons and two risk aversion levels corresponding respectively to the global minimum variance portfolio and the portfolio with a goal of 5% expected return.[12]

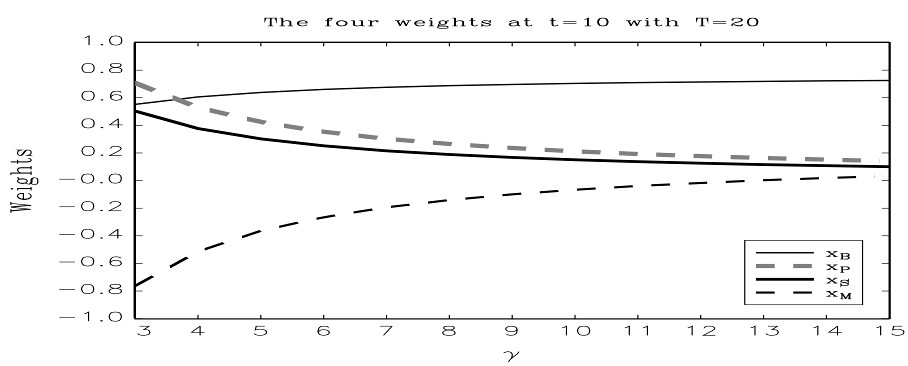

Figure 18: Weights for maturity of 20 years evaluated at time t=10 years

[12]This is also the case for Pagliari (2017) at least for private real estate.

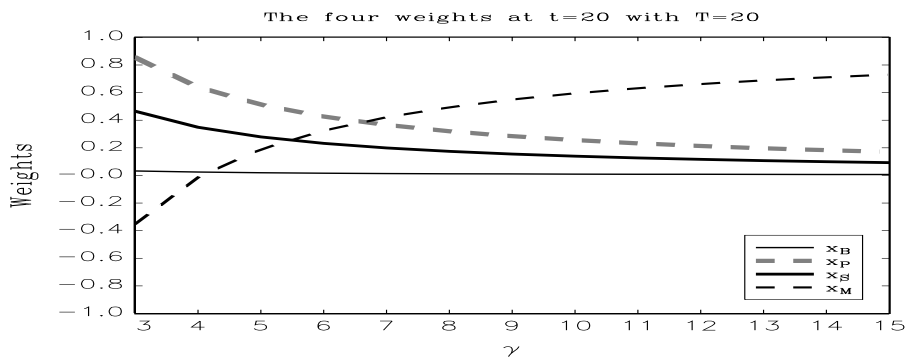

Figure 19: Weights for maturity of 20 years evaluated at time t=20 years

To summarize previous findings, first recall that our main purpose is to propose a continuous-time optimization framework that takes account of mean reverting properties as the discrete-time VaR models of Campbell and Viceira (2002, 2005), Rehring (2012) and Pagliari (2017), while, (contrary to these previous results) allowing to get explicit optimal allocations, in particular as functions of the risk aversion and the time horizon. This is illustrated in previous figures. Second, when illustrating our model on Rehring's data, we are able to recover similar results, at least for the two cases investigated in Rehring's data, namely a high risk aversion and an objective corresponding to 5% expected return. In particular, we show that the allocation on real estate asset is increasing with respect to time while it is not the case for the stock. As mentioned previously, this is mainly due to increasing expected return per year for the real estate asset while it is decreasing over horizon for stocks. Finally, note that contrary to Rehring (2012) and Pagliari (2017), our model provides dynamic optimal solutions for all time horizons and risk aversion levels.

5 Conclusion

Real estate has traditionally been viewed as a low-risk asset class with good diversification properties. Given that most institutional investors have very long-time horizons, typical portfolio optimizations based only on short-term returns, without accounting for return predictability, will produce results quite different from observed optimal allocation for these investors. The result of mean reversion is that the risk of real estate investment would be significantly reduced for long-term investors than for those with shorter investment horizons. This paper emphasizes the impact of the term structure of the asset returns, in particular the role of real estate market volatility on the optimal mix asset allocation. For this purpose, the investor is assumed to maximize the expected utility of her final wealth. We analyze in particular the optimal weights according to mean reverting assumptions about the term structure of the real estate asset. Our results show how optimal portfolio weights depend crucially on the investor's risk aversion and time horizon and also on the considered term structure of asset returns. As emphasized by Feldman (2003) and Rehring (2012), the real estate allocation varies very significantly according to both risk aversion and portfolio horizon. We show also that it can be increasing with respect to current time and portfolio horizon. Finally, our findings allow to emphasize that, by considering mean reverting properties of asset returns, we get in particular real estate portfolio allocations closer to reality for long term investors having significant relative risk aversions, since the optimal weight on real estate asset lies between 10% and 20%. A possible extension would be to examine the effects of market incompleteness[13] as for instance in Karatzas *et al.* (1991) and specific constraints on portfolio weights as in Cvitanic and Karatzas (1992).[14] In that case, it would be possible to determine the associated compensated variation, due to this lack of hedging against real asset risk, as computed for other market frictions in de Palma and Prigent (2009).

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[13]For example, if we assume that the real estate asset dynamics is driven by more than one Brownian motion if we consider multi factor modelling.

[14]Obviously, other models can be introduced and examined. For example, we can take account of stochastic market prices of risk, of labor income as in El Karoui and Jeanblanc (1998), for no traded asset such as in Lioui and Poncet (2001). Nevertheless, note that, even in complete markets, Monte Carlo simulations are often necessary to compute optimal portfolios as for example in Detemple *et al.* (2003) or in Cvitanic *et al.* (2003).

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Appendix

Appendix A: Properties of the numeraire portfolio

Recall that the Radon-Nikodym density of the risk-neutral probability is given by:

$$\eta_t = \exp\left[-\mathbf{M}_t - \int_0^t A(s)ds\right]$$

where

$$\mathbf{M}_t = \int_0^t \lambda_{r,s} dW_s^r + \int_0^t \lambda_{P,s} dW_s^P + \int_0^t \lambda_{S,s} dW_s^S,$$

and

$$A(t) = \frac{1}{2}\left(\lambda_{r,t}^2 + \lambda_{P,t}^2 + \lambda_{S,t}^2\right) + \lambda_{r,t}\lambda_{P,t}\rho^{r,P} + \lambda_{r,t}\lambda_{S,t}\rho^{r,S} + \lambda_{P,t}\lambda_{S,t}\rho^{P,S}.$$

The numeraire portfolio H is equal to:

$$H_t = \frac{\exp\left(\int_0^t r_s ds\right)}{\eta_t}.$$

Consequently, we get:

$$\ln H_t = \int_0^t r_s \; ds + \mathbf{M}_t + \int_0^t A(s)ds.$$

Since $\left(W^r, W^P, W^S\right)$ is a Gaussian process, we can deduce that $\ln H_t$ is Gaussian itself. We have to determine its expectation and variance. More precisely, we determine the conditional expectations for powers z of $\mathbb{E}_t\left[\frac{H_T^z}{H_t^z}\right]$. The ratio $\frac{H_T^z}{H_t^z}$ satisfies:

$$\frac{H_T^z}{H_t^z} = \exp\left[N_z(t,T)\right],$$

with

$$N_z(t,T) = z\left[\int_t^T r_s ds + \mathbf{M}_T - \mathbf{M}_t + \int_t^T A(s)ds\right].$$

The process $N_z(t,T)$ is Gaussian distributed. Thus, to determine $\mathbb{E}_t\left[\frac{H_T^z}{H_t^z}\right]$, we have just to compute $\mathbb{E}_t\left[N_z(t,T)\right]$ and $Var_t\left[N_z(t,T)\right]$, since we have:

$$\mathbb{E}_t\left[\frac{H_T^z}{H_t^z}\right] = \exp\left[\mathbb{E}_t\left[N_z(t,T)\right] + \frac{1}{2}Var_t\left[N_z(t,T)\right]\right].$$

1) Determination of $\mathbb{E}_t\left[N_z(t,T)\right]$. We have:

$$\mathbb{E}_t\left[N_z(t,T)\right] = z\left(\int_t^T \mathbb{E}_t\left[r_u\right] \; du + \int_t^T A(s)ds\right).$$

We deduce: for $u > t$,

$$\mathbb{E}_t\left[r_u\right] = \overline{r} + (r_t - \overline{r})e^{-(u-t)k_r}.$$

Finally, we get:

$$\mathbb{E}_t\left[N_z(t,T)\right] = z(T-t)\left(\overline{r} + (r_t - \overline{r})\frac{\beta_r(T-t)}{T-t} + \frac{1}{T-t}\int_t^T A(s)ds\right),$$

which is equivalent to:

$$\mathbb{E}_t\left[N_z(t,T)\right] = z\Phi_{(t,T)},$$

with:

$$\Phi_{(t,T)} = (T-t)\left[\overline{r} + (r_t - \overline{r})\frac{\beta_r(T-t)}{(T-t)} + \frac{1}{T-t}\int_t^T A(s)ds\right].$$

<u>2) Determination of $Var_t\left[N_z(t,T)\right]$</u>. Step 1: computation of $Var_t\left[\int_t^T r_s ds\right]$ - To calculate $Var_t\left[\int_t^T r_s ds\right]$, we use the Fubini's property for stochastic integrals. We have:

$$r_t = \overline{r} + (r_s - \overline{r})e^{-(t-s)k_r} + a_r e^{-(t-s)k_r}\int_s^t e^{(u-s)k_r}dW_u^r.$$

$$\int_t^T r_u\ du = \overline{r}(T-t) + (r_t - \overline{r})\beta_r(T-t)$$
$$+\int_t^T\left(a_r e^{-(u-t)k_r}\int_t^u e^{(v-t)k_r}dW_v^r\right)du.$$

Then:

$$\begin{aligned}\int_t^T\left[a_r e^{-(u-t)k_r}\int_t^u e^{(s-t)k_r}dW_s^r\right]du &= \int_t^T\left(\int_s^T a_r e^{-(u-t)k_r}e^{(s-t)k_r}du\right)dW_s^r,\\ &= \int_t^T\left(a_r\left[\frac{1-e^{-(T-s)k_r}}{k_r}\right]\right)dW_s^r,\end{aligned}$$

$$Var_t\left[\int_t^T r_s\ ds\right] = a_r^2\int_t^T\left[\frac{1-e^{-(T-s)k_r}}{k_r}\right]^2 ds,$$

$$Var_t\left[\int_t^T r_s\ ds\right] = \frac{a_r^2}{k_r^2}\left[(T-t) + \frac{1}{2k_r}\left[1-\exp(-2(T-t)k_r)\right] - 2\beta_r(T-t)\right].$$

Step 2: computation of $Var_t\left[\mathbf{M}_T - \mathbf{M}_t\right]$:

$$Var_t\left[\mathbf{M}_T - \mathbf{M}_t\right] = 2\int_t^T A(s)ds.$$

Step 3: computation of $Cov_t\left[\int_t^T r_s\ ds;\mathbf{M}_T - \mathbf{M}_t\right]$:

$$Cov_t\left[\int_t^T r_s\ ds;\mathbf{M}_T - \mathbf{M}_t\right] = \int_t^T a_r\left(\lambda_{r,s}+\lambda_{P,s}\rho^{r,P}+\lambda_{S,s}\rho^{r,S}\right)\left[\frac{1-e^{-(T-s)k_r}}{k_r}\right]ds$$

To conclude, recall that:

$$Var_t\left[N_z(t,T)\right] =$$

$$z^2\left(Var_t\left[\int_t^T r_s\ ds\right] + Var_t\left[\mathbf{M}_T - \mathbf{M}_t\right] + 2Cov_t\left[\int_t^T r_s\ ds,\ \mathbf{M}_T - \mathbf{M}_t\right]\right).$$

Therefore, we get:

$$Var_t\left[N_z(t,T)\right] =$$

$$z^2\left(\begin{array}{c}\frac{a_r^2}{k_r^2}\left[(T-t)+\frac{1}{2k_r}\left[1-\exp(-2(T-t)k_r)\right]-2\beta_r(T-t)\right]\\ +2\int_t^T A(s)ds + 2\int_t^T a_r\left(\lambda_{r,,s}+\lambda_{P,s}\rho^{r,P}+\lambda_{S,s}\rho^{r,S}\right)\left[\frac{1-e^{-(T-s)k_r}}{k_r}\right]ds\end{array}\right).$$

Appendix B: Optimal weights

In what follows, we consider the CRRA case. At time t, the optimal CRRA portfolio value is given by:

$$V_T^{CRRA} = \left(\frac{V_0}{\mathbb{E}[H_T^{(\frac{1-\gamma}{\gamma})}]}\right)H_T^{(\frac{1}{\gamma})}. \tag{34}$$

Denote

$$\widetilde{A} = \left(\frac{V_0}{\mathbb{E}[H_T^{(\frac{1-\gamma}{\gamma})}]}\right).$$

To compute the optimal weights x_C, x_B, x_P, and x_S, first we calculate V_t^{CRRA}. We have:

$$V_t^{CRRA}\frac{1}{H_t} = \mathbb{E}_t\left[\frac{1}{H_T}V_T^{CRRA}\right] = \widetilde{A}H_t^{(\frac{1}{\gamma})}\mathbb{E}_t\left[(H_T/H_t)^{(\frac{1}{\gamma}-1)}\right].$$

which implies

$$V_t^{CRRA} = \widetilde{A}H_t^{(\frac{1}{\gamma})}\mathbb{E}_t\left[(H_T/H_t)^z\right] \text{ with } z = \frac{1}{\gamma}-1.$$

Using Relation (25), we get:

$$V_t^{CRRA} = \widetilde{A}H_t^{(\frac{1}{\gamma})}\exp\left[z\Phi_{(t,T)}+\frac{1}{2}z^2\Psi_{(0,T)}\right] \text{ with } z = \frac{1}{\gamma}-1.$$

Then, we determine $\frac{dV_t^{CRRA}}{V_t^{CRRA}}$ by using Relation (34). Applying Ito's formula, we get the martingale part of $\frac{dV_t^{CRRA}}{V_t^{CRRA}}$:

$$\begin{aligned} dV_t^{CRRA} &= (...)dt \\ &+\frac{1}{\gamma}V_t^{CRRA}\left(\lambda_{r,t}dW_t^r+\lambda_{P,t}dW_t^P+\lambda_{S,t}dW_t^S\right) \\ &+V_t^{CRRA}z\left[\beta_r a_r dW_t^r\right] \end{aligned}$$

We obtain the following system:

$$\begin{cases} \frac{\lambda_{r,t}}{\gamma}+za_r\beta_r(T-t) & =\delta_r(t)= & -x_B(t)\beta_r(D_B)a_r \\ \frac{\lambda_{P,t}}{\gamma} & =\delta_P(t)= & x_P(t)\sigma_t^P \\ \frac{\lambda_{S,t}}{\gamma} & =\delta_S(t)= & x_S(t)\sigma_t^S \end{cases}$$

Therefore, we deduce that, at any time t, the weights are given by:

$$\begin{cases} x_B^{CRRA}(t) & = & -\frac{\delta_i(t)}{\beta_i(D)a_i} \\ x_P^{CRRA}(t) & = & \frac{\delta_P(t)}{\sigma_t^P} \\ x_S^{CRRA}(t) & = & \frac{\delta_S(t)}{\sigma_t^S} \\ x_C^{CRRA}(t) & = & 1-x_B^{CRRA}-x_P^{CRRA}-x_S^{CRRA} \end{cases}$$

Recall that Σ is the matrix:

$$\Sigma=\left[\begin{array}{ccc} -\beta_r(D_B)a_r & 0 & 0 \\ 0 & \sigma_t^P & 0 \\ 0 & 0 & \sigma_t^S \end{array}\right].$$

We have:

$$\left[\begin{array}{c} \frac{\lambda_{r,t}}{\gamma}+za_r\beta_r(T-t) \\ \frac{\lambda_P}{\gamma} \\ \frac{\lambda_S}{\gamma} \end{array}\right] = {}^t\Sigma.\left[\begin{array}{c} x_B^{CRRA}(t) \\ x_P^{CRRA}(t) \\ x_S^{CRRA}(t) \end{array}\right],$$

thus:

$$\left[\begin{array}{c} x_B^{CRRA}(t) \\ x_P^{CRRA}(t) \\ x_S^{CRRA}(t) \end{array}\right] = {}^t\Sigma^{-1}.\left(\left[\begin{array}{c} \frac{\lambda_{r,t}}{\gamma}+za_r\beta_r(T-t) \\ \frac{\lambda_{P,t}}{\gamma} \\ \frac{\lambda_{S,t}}{\gamma} \end{array}\right]\right).$$

Recall that $z=\frac{1}{\gamma}-1$ and that the optimal portfolio for the logarithm case X^{Log} is given by:

$$\left[\begin{array}{c} x_B^{Log}(t) \\ x_P^{Log}(t) \\ x_S^{Log}(t) \end{array}\right] = {}^t\Sigma^{-1}.\left(\left[\begin{array}{c} \lambda_{r,t} \\ \lambda_{P,t} \\ \lambda_{S,t} \end{array}\right]\right).$$

Therefore, we have:

$$X^{CRRA}=\frac{1}{\gamma}X^{Log}+(1-\frac{1}{\gamma})X^{Conservative},$$

where X^{Log} is the optimal portfolio for the logarithm case, and $X^{Conservative}$ corresponds to the optimal portfolio for an infinite relative risk aversion γ. This latter portfolio is given by:

$$\left[\begin{array}{c} x_B^{Conservative}(t) \\ x_P^{Conservative}(t) \\ x_S^{Conservative}(t) \end{array}\right] = {}^t\Sigma^{-1}.\left(\left[\begin{array}{c} -a_r\beta_r(T-t) \\ 0 \\ 0 \end{array}\right]\right).$$

Appendix C (Calibration)

In what follows, we detail how we can calibrate drifts, volatilities and instantaneous correlations of the Brownian motions to respectively expected returns per year, standard deviations per year and assets correlations.[15] Indeed, most of the empirical results about mean reverting asset returns are based on cumulated returns on given time periods $[0,T]$ which are further annualized (see e.g. Campbell and Viceira, 2002, 2005; Rehring, 2012; Pagliari, 2017). Using a continuous-time approach, we have to identify the parameters corresponding to instantaneous variations using those which correspond to annualized characteristics of cumulated returns.

In what follows, we consider two financial assets X and Y defined by:

$$dX_t = X_t(\mu_t^X dt + \sigma_t^X dW_t^X), \quad (35)$$
$$dY_t = Y_t(\mu_t^Y dt + \sigma_t^Y dW_t^Y), \quad (36)$$

where $W_t = \left(W_t^X, W_t^Y\right)_{1\leq i\leq d}$ is a 2-dimensional Brownian motion with correlation matrix $\Sigma_{c,T}$ given by

$$\Sigma_{c,T} = \left[\begin{array}{cc} 1 & \rho_T^{W^X,W^Y} \\ \rho_T^{W^X,W^Y} & 1 \end{array}\right],$$

where $\rho_T^{W^X,W^Y} = \left\langle W_t^X, W_t^Y\right\rangle_t$ is a constant calibrated for a fixed management period $[0,T]$. Denote:

$$\Sigma_t = \left[\begin{array}{cc} \sigma_t^X & 0 \\ 0 & \sigma_t^Y \end{array}\right].$$

Computation of expected returns and variances. Denote $\theta_t^X = \mu_t^X - r_t$. The random variable X_T can be expressed as $X_T = \exp\left[N_T^X\right]$ where N_T^X has a Gaussian distribution with

$$\mathbb{E}\left[N_T^X\right] = \Phi_T^X,$$

where

$$\Phi_T^X = \left[\overline{r}T + (r_0 - \overline{r})\beta_r(T) + \int_0^T \left(\theta_s^X - \frac{1}{2}\left[\sigma_s^X\right]^2\right) ds\right],$$

and

$$Var\left[N_T^X\right] = \Psi_T^X,$$

[15] Detailed proofs are available on request.

with

$$\begin{aligned}\Psi_T^X &= \frac{a_r^2}{k_r^2}\left[T + \frac{1}{2k_r}\left[1-\exp(-2Tk_r)\right] - 2\beta_r(T)\right] \\ &+ \int_0^T \left[\sigma_s^X\right]^2 ds + 2a_r\rho_T^{Wr,W^X}\int_0^T \sigma_s^X\left[\frac{1-e^{-(t-s)k_r}}{k_r}\right] ds\end{aligned}$$

Therefore, we have:

$$\mathbb{E}\left[X_T\right] = X_0 \exp\left[\Phi_T^X + \frac{1}{2}\Psi_T^X\right],$$

and the variance is given by:

$$Variance\left[X_T\right] = X_0^2 \exp\left[2\Phi_T^X + \Psi_T^X\right]\left(\exp\left[\Psi_T^X\right] - 1\right).$$

Calibration to the given return per year $g(T)$ and to the given standard deviation per year $h(T)$. We must have:

$$\begin{aligned}\frac{1}{T}\exp\left[\Phi_T^X + \frac{1}{2}\Psi_T^X\right] &= g(T), \\ \frac{1}{T}\exp\left[2\Phi_T^X + \Psi_T^X\right]\left(\exp\left[\Psi_T^X\right] - 1\right) &= h(T),\end{aligned}$$

from which, we deduce:

$$\begin{aligned}\Phi_T^X &= Log\left[Tg(T)/\sqrt{1+\frac{h(T)}{Tg^2(T)}}\right], \\ \Psi_T^X &= Log\left[1+\frac{h(T)}{Tg^2(T)}\right].\end{aligned}$$

Therefore, we have:

$$\begin{aligned}\text{For all } T,\ \frac{\partial \Phi_T^X}{\partial T} &= \frac{\partial}{\partial T} Log\left[Tg(T)/\sqrt{1+\frac{h(T)}{Tg^2(T)}}\right], \\ \frac{\partial \Psi_T^X}{\partial T} &= \frac{\partial}{\partial T} Log\left[1+\frac{h(T)}{Tg^2(T)}\right].\end{aligned}$$

Consequently, we get a first relation:

$$\begin{aligned}\frac{\partial \Phi_T^X}{\partial T} &= \frac{\partial}{\partial T}\left[\overline{r}T + (r_0 - \overline{r})\beta_r(T)\right] + \left(\theta_T^X - \frac{1}{2}\left[\sigma_T^X\right]^2\right), \\ &= \frac{\partial}{\partial T} Log\left[Tg(T)/\sqrt{1+\frac{h(T)}{Tg^2(T)}}\right],\end{aligned}$$

which yields to:

$$\left(\theta_T^X - \frac{1}{2}\left[\sigma_T^X\right]^2\right) =$$

$$\frac{\partial}{\partial T} Log\left[Tg(T)/\sqrt{1+\frac{h(T)}{Tg^2(T)}}\right] - \frac{\partial}{\partial T}\left[\overline{r}T + (r_0 - \overline{r})\beta_r(T)\right]$$

To get the standard deviation, we use the following second relation:

$$\begin{aligned}
\frac{\partial \Psi_T^X}{\partial T} &= \frac{\partial}{\partial T}\left(\frac{a_r^2}{k_r^2}\left[T + \frac{1}{2k_r}\left[1-\exp(-2Tk_r)\right] - 2\beta_r(T)\right]\right) \\
&\quad + \left[\sigma_T^X\right]^2 + 2a_r \frac{\partial}{\partial T}\left(\rho_T^{Wr,W^X}\int_0^T \sigma_s^X\left[\frac{1-e^{-(t-s)k_r}}{k_r}\right]ds\right) \\
&= \frac{\partial}{\partial T} Log\left[1+\frac{h(T)}{Tg^2(T)}\right]
\end{aligned}$$

Finally, we get:

$$\begin{aligned}
&\left[\sigma_T^X\right]^2 + 2a_r \frac{\partial}{\partial T}\left(\rho_T^{Wr,W^X}\int_0^T \sigma_s^X\left[\frac{1-e^{-(T-s)k_r}}{k_r}\right]ds\right) \\
= &\frac{\partial}{\partial T} Log\left[1+\frac{h(T)}{Tg^2(T)}\right] - \frac{\partial}{\partial T}\left(\frac{a_r^2}{k_r^2}\left[T + \frac{1}{2k_r}\left[1-\exp(-2Tk_r)\right] - 2\beta_r(T)\right]\right),
\end{aligned}$$

where we have:

$$\frac{\partial}{\partial T}\left(\rho_T^{Wr,W^X}\int_0^T \sigma_s^X\left[\frac{1-e^{-(T-s)k_r}}{k_r}\right]ds\right) =$$

$$\frac{\partial}{\partial T}\left(\rho_T^{Wr,W^X}\right)\int_0^T \sigma_s^X\left[\frac{1-e^{-(T-s)k_r}}{k_r}\right]ds + \rho_T^{Wr,W^X}\frac{\partial}{\partial T}\left(\int_0^T \sigma_s^X\left[\frac{1-e^{-(T-s)k_r}}{k_r}\right]ds\right).$$

Computation of covariances and correlations. In what follows, we search to calibrate the correlations of the three Brownian motions, namely the three parameters ${\rho_T}^{r,P}$, ${\rho_T}^{r,S}$ and ${\rho_T}^{P,S}$ to Rehring (2012) data (annualized correlations of cumulated asset returns). For this purpose, let us denote:

$$\begin{aligned}
A_t &= E\left[\int_0^t r_s ds\right] = \overline{r}t + (r_0 - \overline{r})\beta_r(t); \\
B_t &= \frac{a_r^2}{k_r^2}\left[t + \frac{1}{2k_r}\left[1-\exp(-2tk_r)\right] - 2\beta_r(t)\right]; \\
C_t &= a_r \rho_T^{Wr,W^X}\int_0^t \sigma_s^X\left[\frac{1-e^{-(t-s)k_r}}{k_r}\right]ds; \\
D_t &= a_r \rho_T^{Wr,W^Y}\int_0^t \sigma_s^Y\left[\frac{1-e^{-(t-s)k_r}}{k_r}\right]ds.
\end{aligned}$$

Taking account of the interest rate randomness, the covariance of asset prices X and Y is given by:

$$Cov_T^{X,Y} =$$

$$X_0 Y_0 \exp\left[\int_0^T \left(\theta_s^X + \theta_s^Y\right) ds + (2A_T + B_T + C_T + D_T)\right]$$
$$\left(\exp\left[\int_0^T \sigma_s^X \sigma_s^Y \rho_T^{W^X,W^Y} ds + (B_T + C_T + D_T)\right] - 1\right).$$

Taking account of the interest rate randomness, the correlation $\rho_T^{X,Y}$ of asset prices X and Y is given by:

$$\rho_T^{X,Y} = Cov_T^{X,Y} / (\sqrt{variance\,(X_T)}\sqrt{variance\,(Y_T)}),$$

which yields to:

$$\rho_T^{X,Y} =$$

$$\frac{\exp\left[(B_T + C_T + D_T)\right]\left(\exp\left[\rho_T^{W^X,W^Y}\int_0^T \sigma_s^X \sigma_s^Y ds + (B_T + C_T + D_T)\right] - 1\right)}{\sqrt{\left(\exp\left[\int_0^T \left[\sigma_s^X\right]^2 ds\right] - 1\right)}\sqrt{\left(\exp\left[\int_0^T \left[\sigma_s^Y\right]^2 ds\right] - 1\right)}}.$$

Thus, we get:

$$\rho_T^{W^X,W^Y} = \frac{1}{\int_0^T \left(\sigma_s^X \sigma_s^Y\right) ds} \times$$

$$\left(\begin{array}{c} Log\left[\dfrac{1 + \rho_T^{X,Y} \exp\left[-\left(B_T + C_T + D_T\right)\right]}{\sqrt{\left(\exp\left[\int_0^T \left[\sigma_s^X\right]^2 ds\right] - 1\right)}\sqrt{\left(\exp\left[\int_0^T \left[\sigma_s^Y\right]^2 ds\right] - 1\right)}}\right] \\ -\left(B_T + C_T + D_T\right) \end{array}\right).$$

Note that, if r_t were deterministic, then we would have a special case with $\mu_s^X = r_s + \theta_s^X$, $\mu_s^Y = r_s + \theta_s^Y$ and $B_t = C_t = D_t = 0$. In such a case, the covariance of asset prices X and Y is given by:

$$\sigma_T^{X,Y} = X_0 Y_0 \exp\left[\int_0^T \left(\mu_s^X + \mu_s^Y\right) ds\right]\left(\exp\left[\rho_T^{W^X,W^Y}\int_0^T \sigma_s^X \sigma_s^Y ds\right] - 1\right)$$

Therefore, for a fixed management period $[0, T]$ and for a given correlation function $\rho_T^{X,Y}$ defined by:

$$\rho_T^{X,Y} = Cov_T^{X,Y} / (\sqrt{variance\,(X_T)}\sqrt{variance\,(Y_T)}),$$

we find:

$$\rho_T^{X,Y} =$$

$$\frac{\left(\exp\left[\rho_T^{W^X,W^Y}\int_0^T\sigma_s^X\sigma_s^Y\,ds\right]-1\right)}{\sqrt{\left(\exp\left[\int_0^T[\sigma_s^X]^2\,ds\right]-1\right)}\sqrt{\left(\exp\left[\int_0^T[\sigma_s^Y]^2\,ds\right]-1\right)}}.$$

Thus, we get:

$$\rho_T^{W^X,W^Y}=\frac{1}{\int_0^T\sigma_s^X\sigma_s^Y\,ds}\times$$

$$Log\left[1+\rho_T^{X,Y}\sqrt{\left(\exp\left[\int_0^T[\sigma_s^X]^2\,ds\right]-1\right)}\sqrt{\left(\exp\left[\int_0^T[\sigma_s^Y]^2\,ds\right]-1\right)}\right].$$

The previous formula can be applied not only to the real estate-stock case but also, when the bond is involved. However, as detailed in Section 2, the bond modelling is such that the term structure is affine (see Relation 7).